NP-completeness proof by reduction
adapted from CLRS §34.5.4

CS 758/858 F22 Recitation
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1 Objective

We will show that the traveling-salesperson problem (TSP) is NP-complete by showing that TSP $\in$ NP and that TSP is NP-hard.

2 Problem formalization

TSP $= \{\langle G, c, k\rangle :$

$G = (V, E)$ is a complete graph,

c is a function from vertices to non-negative integers,

k is a non-negative integer, and

G has a TSP tour with cost at most $k\}$.

3 TSP $\in$ NP

We want to show that TSP $\in$ NP by showing that a solution to TSP is verifiable is polynomial time.

Let a certificate for TSP be the sequence of vertices in a tour. The verification algorithm must:

- check that each vertex in the graph is included exactly once in the sequence (except coming home to the start vertex).
- add up the total cost of edges in the tour
- check that the tour cost is no greater than k.

This verification can be performed in polynomial time, since we are simple iterating through the certificate, and at each edge adding its cost to the sum and at each vertex incrementing our count of how many times we've visited it; at the end, we iterate through the vertices to make sure their visit counts are all exactly 1. This runs in $\Theta(n)$ time, so is obviously polytime.

4 TSP is NP-hard

We want to show that TSP is NP-hard by showing that there is a polynomial time reduction from Ham-Cycle (HC), which is known to be NP-complete.

4.1 Reduction algorithm

We want to show that, given any instance of HC, we can construct an equivalent instance of TSP.

Given an instance $G = (V, E)$ of HC, we construct an instance of TSP by

- forming the complete graph $G' = (V, E')$; that is, we're adding edges between all vertices, but no self-loop edges. (Formally, $E' = \{(i, j) : i, j \in V, i \neq j\}$.)
- defining the cost function $c(e)$ of each edge $e \in E'$ is 0 if $e \in E$ and 1 if $e \notin E$.

The instance of TSP (reduced from HC) is thus $\langle G', c, 0 \rangle$.

4.2 Reduction correctness

We want to show that an instance G of HC has a hamiltonian path if and only if G' (the graph in the corresponding TSP instance, by the reduction above) has a tour of cost at most zero. We need to prove the implication in each direction.

4.2.1 Given instance of HC, show that its reduction is instance of TSP

Suppose G is an instance of HC, with its path H. Each edge of H is in E, and therefore has cost zero in G'. Therefore H is a tour in G' with zero cost.

4.2.2 Given instance of TSP, show that its reduction is instance of HC

Suppose G' from an instance of TSP has a tour T of cost zero. Then all T's edges must be in E. Therefore T is a hamiltonian path in G.

4.3 Reduction analysis

We want to show that this reduction is polytime.

Adding the edges to make the complete graph is $O(V^2)$, and so is iterating through all those edges to set their costs, so the reduction is polytime.

5 Conclusion

We have shown that TSP is in NP, and that TSP is NP-hard because there exists a polytime reduction from another, known-NPC problem (Ham-Cycle). Therefore, TSP is NP-complete.