http://www.cs.unh.edu/~ruml/cs758
Red-Black Trees

- Searching
- Balanced Trees
- Red-Black Trees
- Rotation
- Insert($z$)
- Fixing Insertion
- Fixup Invariant
- Fix-insert($z$)
- Termination
- Break

Red-Black Trees
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- Insert($z$)
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- Break
1. AVL Trees (1962)
2. 2-3 Trees
3. red-black trees (1972, popularized 1978)
4. AA trees (1992)
5. left-leaning red-black trees (2008)

probabilistically balanced

1. treaps
2. skip lists
node: data, left, right, parent, color

1. every node is either red or black
2. the root is black
3. (consider nil to be black)
4. both children of a red node are black
5. from any node, all paths to leaves have the same ‘black height’

search and traversal are unchanged
useful subroutines:
- rotate-right
- rotate-left
**Insert(z)**

1. $z$’s parent $\leftarrow$ find-parent($z$, root, nil)
2. if parent is nil
3. root $\leftarrow z$
4. else
5. if $z$ should be before parent
6. parent’s left child $\leftarrow z$
7. else
8. parent’s right child $\leftarrow z$
9. $z$’s children $\leftarrow$ nil
**Insert(z)**

1. \(z\)’s parent ← \text{find-parent}(z, \text{root}, \text{nil})
2. if parent is nil
3. root ← \(z\)
4. else
5. if \(z\) should be before parent
6. parent’s left child ← \(z\)
7. else
8. parent’s right child ← \(z\)
9. \(z\)’s children ← nil
10. color \(z\) \text{ red}
11. fix-insert(\(z\))
Fixing Insertion

Recall properties:

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Cases:

1. red root (property 2)
2. two red in a row (property 4)
Fixup Invariant

Cases:
1. red root (property 2)
2. two red in a row (property 4)

During fixup:
1. \( z \) is red
2. if \( z \)'s parent is the root, it is black
3. at most, property 2 xor 4 is violated at \( z \)
   (a) if 2: because \( z \) is root and red
   (b) if 4: because \( z \) and parent are red
Fixup Invariant

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Initialization:
1. we colored \( z \) red
2. we didn’t touch \( z \)’s parent, and roots are black
3. just saw this
Fix-insert($z$)

1. while $z$’s parent is red
2. if $z$’s parent is a left child
3. $y \leftarrow z$’s uncle (a right child)
4. if $y$ is red
5. color $z$’s parent black \hspace{1cm} \textit{case 1}
6. color $z$’s uncle $y$ black
7. color $z$’s grandparent red
8. $z \leftarrow z$’s grandparent
9. else if $z$ is a right child
10. $z \leftarrow z$’s parent \hspace{1cm} \textit{case 2}
11. rotate-left($z$)
12. color $z$’s parent black \hspace{1cm} \textit{case 3}
13. color $z$’s grandparent red
14. rotate-right($z$’s grandparent)
15. else, 3 symmetric cases (left$\leftrightarrow$right)
16. color root black
Assuming other properties are maintained, are we red-black now?

Leverage the invariant:

1. irrelevant
2. irrelevant
3. only 2 xor 4 can be violated in loop
   (a) if 2: root colored black at end, so 2 not violated
   (b) if 4: z’s parent now black, so 4 not violated
Assuming other properties are maintained, are we red-black now?

Leverage the invariant:
1. irrelevant
2. irrelevant
3. only 2 xor 4 can be violated in loop
   (a) if 2: root colored black at end, so 2 not violated
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How to make progress around loop while maintaining invariant?
asst 3
asst 4
piazza, grading
Red-Black Trees

- Maintenance
- Case 1
- Case 2
- Case 3
- Complexity
- EOLQs

Red-Black Trees
central problem: prop 4 violated: \( z \) and parent are red

note \( z \) has an uncle because the root is black

3 cases (+ 3 more by symmetry of \( z \)’s parent being left/right):
1. \( z \)’s uncle \( y \) is also red (we have a red layer)
2. \( z \)’s uncle \( y \) is black and \( z \) is right child
3. \( z \)’s uncle \( y \) is black and \( z \) is left child
central problem: prop 4 violated: \( z \) and parent are red

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Plan:

1. fix case 1, possibly introducing case 2.
2. reduce case 2 to case 3.
3. fix case 3.
Case 1

case 1: $z$’s uncle $y$ is also red

solution: move redness up

1. color $z$’s parent and uncle black
2. color grandparent red and recur

fixup loop invariants:

1. $z$ is red
2. if $z$’s parent is the root, it is black (unchanged)
3. at most, property 2 xor 4 is violated at new $z$. Note previous violations at old $z$ are fixed.

   (a) if 2: because $z$ is root and red
   (b) if 4: because $z$ and parent are red

if new $z$ is root, will be colored black, increasing all heights
Case 2

case 2: $z$’s uncle $y$ is black and $z$ is right child

reduce to case 3: $z$’s uncle $y$ is black and $z$ is left child

rotation doesn’t affect any properties
Case 3

case 3: $z$’s uncle $y$ is black and $z$ is left child

fix prop 4 at $z$: pull blackness down to $z$’s parent and rotate grandparent under it.

fixup loop invariants:

1. $z$ is red
2. if $z$’s parent is the root, it is black
3. at most, property 2 xor 4 is violated at $z$.
   
   (a) can’t be prop 2
   
   (b) if 4: fixed because $z$’s parent is now black
   
   (c) note black-height is preserved!

We are done and loop will exit
finding place is
finding place is $O(\log n)$

one fixup iteration is constant time

fixup loops only when moving up, so is
finding place is $O(\lg n)$

one fixup iteration is constant time

fixup loops only when moving up, so is $O(\lg n)$

how many rotations are performed?
For example:

- What’s still confusing?
- What question didn’t you get to ask today?
- What would you like to hear more about?

Please write down your most pressing question about algorithms and put it in the box on your way out.

*Thanks!*