Risk-Averse Decision Making and Control

Marek Petrik
University of New Hampshire

Mohammad Ghavamzadeh
Adobe Research

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Outline

Schedule

9:00–9:20	**Introduction to risk-averse modeling**
9:20–9:40	Value at Risk and Average Value at Risk
9:40–9:50	*Break*
9:50–10:30	Coherent Measures of Risk: Properties and methods
10:30–11:00	*Coffee break*
11:00–12:30	Risk-averse reinforcement learning
12:30–12:40	*Break*
12:40–12:55	Time consistent measures of risk

Risk Aversion

<u>Risk</u> (Wikipedia):

> ***Risk*** *is the potential of ~~gaining~~ or losing something of value. . . .* ***Uncertainty*** *is a potential, unpredictable, and uncontrollable outcome;* ***risk*** *is a consequence of action taken in spite of uncertainty.*

<u>Risk aversion</u> (Wikipedia):

> *. . .* ***risk aversion*** *is the behavior of humans, when exposed to uncertainty, to attempt to reduce that uncertainty. . . .*

Tutorial: Modern methods for risk-averse decision making

Desire for Risk Aversion

- Empirical evidence:
  1. People buy insurance
  2. Diversifying financial portfolios
  3. Experimental results

- Other reasons:
  - Reduce contingency planning

Where Risk Aversion Matters

- Financial portfolios
- Heath-care decisions
- Agriculture
- Public infrastructure
- Self-driving cars?

When Risks Are Ignored . . .

Housing bubble leads to a financial collapse

Seawalls overflow in a tsunami

"All the News That's Fit to Print"

The New York Times

NEW YORK, THURSDAY, SEPTEMBER 18, 2008

McCain Seen as Less Likely To Bring Change, Poll Finds

Margin in Race About Same as Before Conventions

NEW PHASE IN FINANCE CRISIS AS INVESTORS RUN TO SAFETY

GLOBAL REACTION

Credit Markets Suffer — Borrowing Costs Soar as Result

U.S. Court, a Longtime Beacon, Is Now Guiding Fewer Nations

Need to Quantify Risk

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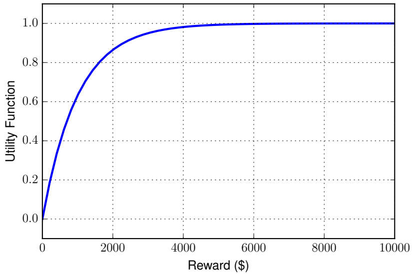
- Expected utility theory:

$$\mathbb{E}[u(X)] = \mathbb{E}[\text{utility}(X)]$$

- Exponential utility function (Bernoulli functions):

$$u(x) = \frac{1 - e^{-a\,x}}{a}$$



Example: Buying Car Insurance

Car value: $\$10\,000$

Insurance options

Option	Deductible	Cost
X_1	$\$10\,000$	$\$0$
X_2	$\$2\,000$	$\$112$
X_3	$\$100$	$\$322$

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Expected utility:

Event	$\mathbb{P}$	X_1	X_2	X_3
No accident	92%	$\$0$	$-\$112$	$-\$322$
Minor accident	7.5%	$-\$2\,500$	$-\$2\,112$	$-\$422$
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Example: Buying Car Insurance

Car value: $\$10\,000$

Insurance options

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X_1	$\$10\,000$	$\$0$
X_2	$\$2\,000$	$\$112$
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Insurance options

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X_2	$\$2\,000$	$\$112$
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$\mathbb{E}$		$-\$\mathbf{237.50}$	$-\$272.00$	$-\$330.00$

Risk-neutral choice: no insurance

Risk Averse Utility Functions

- Exponential utility function

$$u(x) = \frac{1 - \exp(-10^{-6} \cdot (x + 10^5))}{10^{-6}}$$

- X_1 – no insurance
- X_2 – high deductible insurance

Event	$\mathbb{P}$	X_1	$u(X_1)$	X_2	$u(X_2)$
No accident	92%	\$0	1 111	−\$112	1 111
Minor accident	7.5%	−\$2 500	1 109	−\$2 112	1 110
Major accident	0.5%	−\$10 000	0	−\$2 112	1 110
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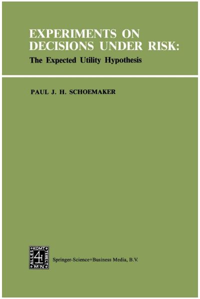
$\mathbb{E}$		−\$237.50	1 105	−\$272.00	**1 111**

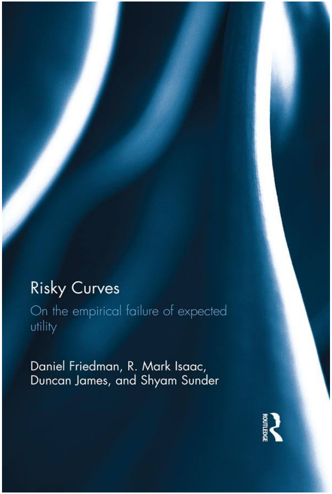
Prefer insurance, but difficult to interpret and elicit

Drawbacks of Expected Utility Theory



(Schoemaker 1980)

1. Does not explain human behavior
2. Difficult to elicit utilities
3. Complicates optimization



(Friedman et al. 2014)

Major Alternatives for Measuring Risk

1. **Markowitz portfolios**: Penalize dispersion risk

$$\begin{aligned} \min_{c \geq \mathbf{0}} \quad & \operatorname{Var}\Big[\sum_i c_i \cdot X_i\Big] \\ \text{s.t.} \quad & \mathbb{E}\Big[\sum_i c_i \cdot X_i\Big] = \mu, \quad \sum_i c_i = 1 \end{aligned}$$

Limited modeling capability and also penalizes upside

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   Limited modeling capability and also penalizes upside

2. **Risk measures**: (Artzner et al. 1999)
   - Value at risk (V@R)
   - Conditional value at risk (CV@R)
   - Coherent measures of risk

Coherent Measures of Risk

Topic of this tutorial

- Alternative to expected utility theory

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+ Flexible modeling framework

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Coherent Measures of Risk

Topic of this tutorial

- Alternative to expected utility theory
- \+ Flexible modeling framework
- \+ Convenient to use with optimization and decision making
- \+ Easier to elicit than utilities
- − Difficulties in sequential decision making

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Risk Measure

Risk measure: function ρ that maps random variable to a real number

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$$\rho(X) = \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

  - Risk neutral

Risk Measure

Risk measure: function ρ that maps random variable to a real number

- **Expectation** is a risk measure

$$\rho(X) = \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

  - Risk neutral
- **Worst-case** is a risk measure

$$\rho(X) = \min[X] = \min_{\omega \in \Omega} X(\omega)$$

  - Very risk averse

V@R: Value at Risk

$$\rho(X) = \mathrm{V@R}_{\alpha}(X) = \sup\left\{t \;:\; \mathbb{P}[X \le t] < \alpha\right\}$$

Rewards smaller than $\mathrm{V@R}_{\alpha}(X)$ with probability at most α

Example α values:

$\alpha = 0.5$ Median

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- $\alpha = 0.3$ More conservative

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$\alpha = 0.3$ More conservative

$\alpha = 0.05$ Conservative

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Example α values:

$\alpha = 0.5$ Median

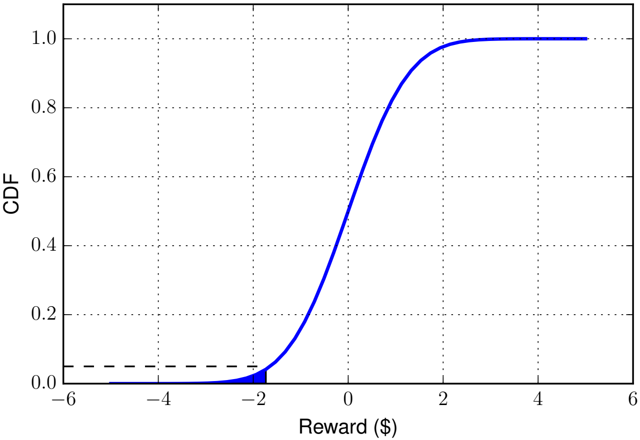
$\alpha = 0.3$ More conservative

$\alpha = 0.05$ Conservative

$\alpha = 0$ Worst case

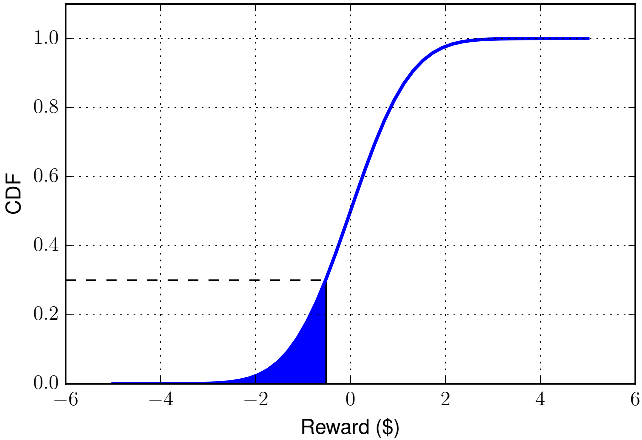
V@R Example 1: Cumulative Distribution Function

$$\text{V@R}_{0.05}(X) = -1.7$$



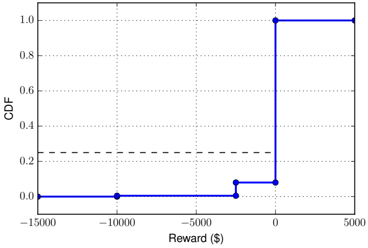
V@R Example 2: Cumulative Distribution Function

$$\mathrm{V@R}_{0.3}(X) = -0.5$$



Car Insurance And V@R: 25%

Event	$\mathbb{P}$	X_1
No accident	92%	\$0
Minor accident	7.5%	−\$2 500
Major accident	0.5%	−\$10 000



$$\mathrm{V@R}_\alpha(X) = \sup\left\{t \ : \ \mathbb{P}[X \leq t] < \alpha\right\} \qquad \alpha = 0.25$$

t	$\mathbb{P}[X \leq t]$	α
−\$2 600	0.005	0.25
−\$2 500	0.008	0.25
\$0	1.000	0.25

Car Insurance And V@R: 8%

Event	$\mathbb{P}$	X_1
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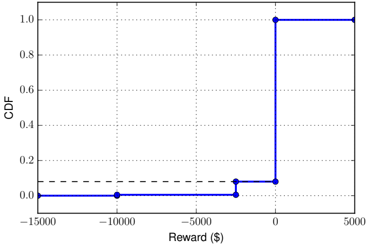
Minor accident	7.5%	−\$2 500
Major accident	0.5%	−\$10 000



$$\mathrm{V@R}_\alpha(X) = \sup\left\{t \;:\; \mathbb{P}[X \le t] < \alpha\right\} \qquad \alpha = 0.008$$

t	$\mathbb{P}[X \le t]$	α
−\$2 500	0.005	0.008
−\$2 400	0.008	0.008

Car Insurance And V@R

- X_1: no insurance (high risk)
- X_2: high deductible insurance (medium risk)
- X_3: low deductible insurance (low risk)

Event	$\mathbb{P}$	X_1	X_2	X_3
No accident	92%	\$0	−\$112	−\$322
Minor accident	7.5%	−\$2 500	−\$2 112	−\$422
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$\mathbb{E}$		−\$**238**	−\$272	−\$330

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$\mathrm{V@R}_{0.25}$		**\$0**	−\$112	−\$322

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$\mathrm{V@R}_{0.25}$		**\$0**	−\$112	−\$322
$\mathrm{V@R}_{0.05}$		−\$2 500	−\$2 112	**−\$422**

Properties of V@R

+ Preserves affine transformations:

$$\mathrm{V@R}_\alpha(\tau \cdot X + c) = \tau \cdot \mathrm{V@R}_\alpha(X) + c$$

+ Simple and intuitive to model and understand
+ Compelling meaning in finance
− Ignores heavy tails
− Not convex

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Coherent measures of risk: Preserve $\mathrm{V@R}$ positives and improve negatives (Artzner et al. 1999)

Average Value at Risk

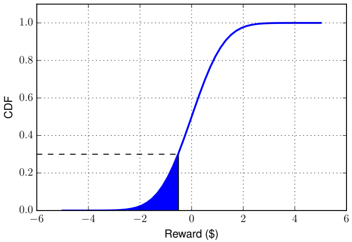
- AKA Conditional Value at Risk and Expected Shortfall
- Popular coherent risk measure ρ
- Simple definition for atomless distributions:

$$\mathrm{CV@R}_\alpha(X) = \mathbb{E}\Big[X \mid X \le \mathrm{V@R}_\alpha(X)\Big]$$

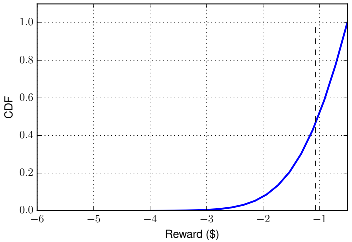
- Recall: $\mathrm{V@R}_\alpha(X) = \sup\big\{t \;:\; \mathbb{P}[X \le t] < \alpha\big\}$
- Convex extension of $\mathrm{V@R}$ (Rockafellar and Uryasev 2000)

V@R vs CV@R: Cumulative Distribution Function

$$\mathrm{V@R}_{0.3}(X) = -0.5$$



$$\mathrm{CV@R}_{0.3}(X) = -1.1$$



CV@R vs V@R: Heavy Tails

A more expensive car?

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Major acc.	0.5%	−\$10 000
$\text{V@R}_{0.05}$		−\$2 500
$\text{CV@R}_{0.05}$		−\$3 250

Event	$\mathbb{P}$	X_1
No accident	92%	\$0
Minor acc.	7.5%	−\$2 500
Major acc.	0.5%	−\$1 000 000
$\text{V@R}_{0.05}$		−\$2 500
$\text{CV@R}_{0.05}$		−\$102 250

V@R: Heavy Tails and Financial Crisis

International Review of Financial Analysis 33 (2014) 33–38

Contents lists available at ScienceDirect

International Review of Financial Analysis

ELSEVIER

Financial crisis, Value-at-Risk forecasts and the puzzle of dependency modeling

CrossMark

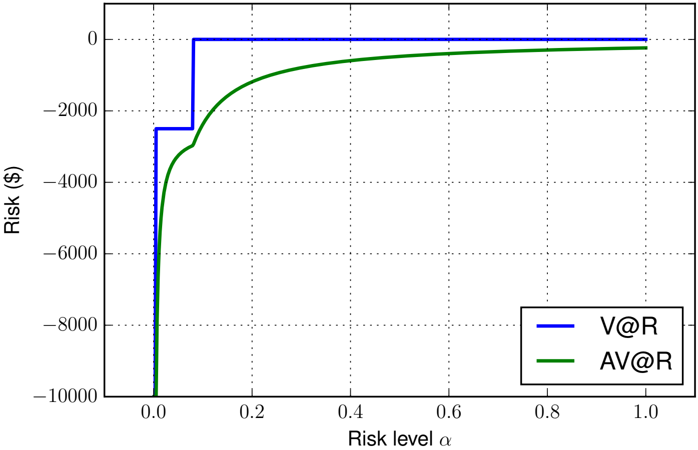
T. Berger *, M. Missong

University of Bremen, Wilhelm-Herbst-Str. 5, 28359 Bremen, Germany

ARTICLE INFO

ABSTRACT

CV@R vs V@R: Continuity



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Coherent Measures of Risk

- Generalize $\mathrm{CV@R}$ to allow more general models
- Framework introduced in (Artzner et al. 1999)
- Coherence: Requirements for risk measure ρ to satisfy
- Our treatment based on (Shapiro, Dentcheva, and Ruszczynski 2009) and (Follmer and Schied 2011)

Coherence Requirements of Risk Measures

1. **Convexity**: (really concavity for maximization!)

$$\rho(t \cdot X + (1-t) \cdot Y) \geq t \cdot \rho(X) + (1-t) \cdot \rho(Y)$$

2. **Monotonicity**:

If $X \succeq Y$, then $\rho(X) \geq \rho(Y)$

3. **Translation equivariance**: For a constant a:

$$\rho(X + a) = \rho(X) + a$$

4. **Positive homogeneity**: For $t > 0$, then:

$$\rho(t \cdot X) = t \cdot \rho(X)$$

Convexity

Why: Diversification should decrease risk (and it helps with optimization)

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Event	$\mathbb{P}$	X_1	X_2	$\frac{1}{2}X_1 + \frac{1}{2}X_2$
No accident	92%	$\$0$	$-\$112$	$-\$56$
Minor accident	7.5%	$-\$2\,500$	$-\$2\,112$	$-\$2\,306$
Major accident	0.5%	$-\$10\,000$	$-\$2\,112$	$-\$6\,056$
CV@R		$-\$238$	$-\$272$	$-\$240$

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Event	$\mathbb{P}$	X_1	X_2	$\frac{1}{2}X_1 + \frac{1}{2}X_2$
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Major accident	0.5%	$-\$10\,000$	$-\$2\,112$	$-\$6\,056$
CV@R		$-\$238$	$-\$272$	$-\$240$

$$-240 \geq \frac{-238 + -272}{2} = -255$$

Monotonicity

Why: Do not prefer an outcome that is always worse

$$\text{If } X \succeq Y, \text{ then } \rho(X) \geq \rho(Y)$$

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Monotonicity

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$$-\$320 \leq -\$238$$

Translation equivariance

Why: Risk is measured in the same units as the reward

$$\rho(X + a) = \rho(X) + a$$

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More expensive insurance by $\$100$

Event	$\mathbb{P}$	X_2	X_2
No accident	92%	$-\$112$	$-\$212$
Minor accident	7.5%	$-\$2\,112$	$-\$2\,212$
Major accident	0.5%	$-\$2\,112$	$-\$2\,212$
ρ		$-\$272$	$-\$372$

Translation equivariance

Why: Risk is measured in the same units as the reward

$$\rho(X + a) = \rho(X) + a$$

More expensive insurance by $\$100$

Event	$\mathbb{P}$	X_2	X_2
No accident	92%	$-\$112$	$-\$212$
Minor accident	7.5%	$-\$2\,112$	$-\$2\,212$
Major accident	0.5%	$-\$2\,112$	$-\$2\,212$
ρ		$-\$272$	$-\$372$

$$-\$372 = -\$272 - \$100$$

Positive homogeneity

Why: Risk is measured in the same units as the reward

$$\rho(t \cdot X) = t \cdot \rho(X)$$

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What if the prices are in €: $\$1 = €0.94$

Event	$\mathbb{P}$	X_2	X_2
No accident	92%	$-\$112$	$-€105$
Minor accident	7.5%	$-\$2\,112$	$-€1\,985$
Major accident	0.5%	$-\$2\,112$	$-€1\,985$
ρ		$-\$272$	$-€256$

Positive homogeneity

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$$\rho(t \cdot X) = t \cdot \rho(X)$$

What if the prices are in €: $\$1 = €0.94$

Event	$\mathbb{P}$	X_2	X_2
No accident	92%	−\$112	−€105
Minor accident	7.5%	−\$2 112	−€1 985
Major accident	0.5%	−\$2 112	−€1 985
ρ		−\$272	−€256

$$-\$272 = -€256$$

Convex Risk Measures

Weaker definition than coherent risk measures

1. **Convexity**:

$$\rho(t \cdot X + (1-t) \cdot Y) \leq t \cdot \rho(X) + (1-t) \cdot \rho(Y)$$

2. **Monotonicity**:

If $X \succeq Y$, then $\rho(X) \geq \rho(Y)$

3. **Translation equivariance**: For a constant a:

$$\rho(X + a) = \rho(X) + a$$

4. ~~Positive homogeneity~~

Additional Property: Law Invariance

Value of risk measure is independent of the names of the events

Consider a coin flip

Event	$\mathbb{P}$	X	Y
Heads	$1/2$	1	0
Tails	$1/2$	0	1

Require that $\rho(X) = \rho(Y)$; violated by some coherent risk measures

<u>Distortion risk measures</u>: coherence & law invariance & comonotonicity

Simple Coherent Measures of Risk

- **Expectation**:

$$\rho(x) = \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

1. Convexity: $\mathbb{E}[X]$ is linear
2. Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \succeq Y$
3. Translation equivariance: $\mathbb{E}[X + a] = \mathbb{E}[X] + a$
4. Positive homogeneity: $\mathbb{E}[t \cdot X] = t \cdot \mathbb{E}[X]$ for $t > 0$

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- **Worst case**:

$$\rho(X) = \min[X] = \min_{\omega \in \Omega} X(\omega)$$

1. Convexity: $\min[X]$ is convex
2. Monotonicity: $\min[X] \geq \min[Y]$ if $X \succeq Y$
3. Translation equivariance: $\min[X + a] = \min[X] + a$
4. Positive homogeneity: $\min[t \cdot X] = t \cdot \mathbb{E}[X]$ for $t > 0$

CV@R for Discrete Distributions

- Simple definition is **not coherent**

$$\mathrm{CV@R}_\alpha(X) = \mathbb{E}\left[X \;\middle|\; X \leq \mathrm{V@R}_\alpha(X)\right]$$

  - Violates **convexity** when distribution has atoms (discrete distributions)

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- **Coherent definition of** $\mathrm{CV@R}$:

$$\mathrm{CV@R}_\alpha(X) = \sup_t \Big\{ t + \frac{1}{\alpha}\mathbb{E}[X - t]_- \Big\}$$

  - $t^\star = \mathrm{V@R}_\alpha(X)$ when the distribution is atom-less

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- Definitions the same for continuous distributions

Computing CV@R

- **Discrete distributions**: Solve a linear program

$$\begin{aligned} \max_{t,y} \quad & t + \frac{1}{\alpha} p^\top y \\ \text{s.t.} \quad & y \leq X - t, \\ & y \leq \mathbf{0} \end{aligned}$$

- **Continuous distributions**: Closed form for many (Nadarajah, Zhang, and Chan 2014; Andreev, Kanto, and Malo 2005)

Car Insurance and CV@R

- X_1 – no insurance
- X_2 – high deductible insurance
- X_3 – low deductible insurance

Event	$\mathbb{P}$	X_1	X_2	X_3
No accident	92%	\$0	−\$112	−\$322
Minor accident	7.5%	−\$2 500	−\$2 112	−\$422
Major accident	0.5%	−\$10 000	−\$2 112	−\$422
$\mathbb{E}$		−\$**238**	−\$272	−\$330

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$\mathrm{V@R}_{0.25}$		**\$0**	−\$112	−\$322
$\mathrm{CV@R}_{0.25}$		−\$950	−\$752	**−\$354**

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$\text{V@R}_{0.05}$		−\$2 500	−\$2 112	−\$**422**
$\text{CV@R}_{0.05}$		−\$3 250	−\$2 112	−\$**422**

Robust Representation of Coherent Risk Measures

- **Important representation for analysis and optimization**
- For any coherent risk measure ρ:

$$\rho(X) = \min_{\xi \in \mathfrak{A}} \mathbb{E}_{\xi}\left[X\right] = \inf_{\xi \in \mathfrak{A}} \xi^{\top} X$$

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- $\mathfrak{A}$ is a set of measures such that is:
  1. convex
  2. bounded
  3. closed
- Proof: Double convex conjugate
  - Convex conjugate:

$$\rho^\star(y) = \sup_x x^\top y - \rho(x)$$

  - Fenchel–Moreau theorem:

$$\rho^{\star\star}(x) = \rho(x)$$

Robust Set for CV@R

$$\mathrm{CV@R}_\alpha(X) = \sup_t \Big\{ t + \frac{1}{\alpha}\mathbb{E}[X - t]_- \Big\}$$

- Robust representation:

$$\rho(X) = \inf_{\xi \in \mathfrak{A}} \mathbb{E}_\xi \big[X \big]$$

- Robust set for probability distribution P:

$$\mathfrak{A} = \Big\{ \xi \geq \mathbf{0} \mid \xi \leq \frac{1}{\alpha} P,\ \mathbf{1}^\top \xi = 1 \Big\}$$

Robust Set for CV@R

- Robust representation:

$$\rho(X) = \min_{\xi \in \mathfrak{A}} \mathbb{E}_\xi \left[X\right]$$

$$\mathfrak{A} = \left\{ \xi \geq \mathbf{0} \mid \xi \leq \frac{1}{\alpha} P,\ \mathbf{1}^\top \xi = 1 \right\}$$

- Random variable: $X = [10, 5, 2]$
- Probability distribution: $p = [{}^1\!/_3, {}^1\!/_3, {}^1\!/_3]$
- $\mathrm{CV@R}_{1/2}(X) =$

$$\min_{\xi \geq \mathbf{0}} \quad 10\,\xi_1 + 5\,\xi_2 + 2\,\xi_3$$

$$\xi_i \leq \frac{1}{\alpha} p_i = \frac{1}{1/2} {}^1\!/_3 = \frac{2}{3} \qquad \xi_1 + \xi_2 + \xi_3 = 1$$

Other Coherent Risk Measures

1. Combination of expectation and $\mathrm{CV@R}$
2. Entropic risk measure
3. Coherent entropic risk measure (convex, incoherent)
4. Risk measures from utility functions
5. ...

Convex Combination of Expectation and CV@R

- CV@R ignores the mean return
- Risk-averse solutions bad in expectation
- Practical trade-off: Combine mean and risk

$$\rho(X) = c \cdot \mathbb{E}[X] + (1-c) \cdot \mathrm{CV@R}_\alpha(X)$$

Entropic Risk Measure

$$\rho(X) = -1/\tau \ln \mathbb{E}\left[e^{-\tau \cdot X}\right] \quad \tau > 0$$

- **Convex risk measure**

Entropic Risk Measure

$$\rho(X) = -{}^{1}\!/_{\tau} \ln \mathbb{E}\left[e^{-\tau \cdot X}\right] \quad \tau > 0$$

- **Convex risk measure**
- Incoherent (violates translation invariance)
- No robust representation

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$$\rho(X) = -{}^{1}\!/_{\tau} \ln \mathbb{E}\left[e^{-\tau \cdot X}\right] \quad \tau > 0$$

- **Convex risk measure**
- Incoherent (violates translation invariance)
- No robust representation
- **Coherent entropic risk measure**: (Föllmer and Knispel 2011)

$$\rho(X) = \max_{\xi \geq \mathbf{0}} \left\{ \mathbb{E}_{\xi}[X] \mid KL(\xi \mid P) \leq c, \mathbf{1}^{\top}\xi = 1 \right\}$$

Risk Measure From Utility Function

- Concave utility function $u(\cdot)$
- Construct a **coherent** risk measure from g?

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$$\rho(X) = \mathbb{E}[u(X)]$$

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Risk Measure From Utility Function

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- Construct a **coherent** risk measure from g?
- **Direct construction**:

$$\rho(X) = \mathbb{E}[u(X)]$$

  Not coherent or convex
- **Optimized Certainty Equivalent** (Ben-Tal and Teboulle 2007)

$$\rho(X) = \sup_t \left(t + \mathbb{E}[g(X - t)]\right)$$

Optimized Certainty Equivalent

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- How much consume now given uncertain future

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- **Convex** risk measure for any concave u
- **Coherent** risk measure for pos. homogeneous u

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- Exponential u: OCE = entropic risk measure
- Piecewise linear u: OCE = $\mathrm{CV@R}$

Recommended References

- Lectures on Stochastic Programming: Modeling and Theory (Shapiro, Dentcheva, and Ruszczynski 2014)
- Stochastic Finance: An Introduction in Discrete Time (Follmer and Schied 2011)

Remainder of Tutorial: Multistage Optimization

- How to apply risk measures when optimizing over multiple time steps
- Results in machine learning and reinforcement learning
- Time or dynamic consistency in multiple time steps

Schedule

9:00–9:20	Introduction to risk-averse modeling
9:20–9:40	Value at Risk and Average Value at Risk
9:40–9:50	*Break*
9:50–10:30	Coherent Measures of Risk: Properties and methods
10:30–11:00	***Coffee break***
11:00–12:30	Risk-averse reinforcement learning
12:30–12:40	*Break*
12:40–12:55	Time consistent measures of risk

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Please see the other slide deck

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12:40–12:55	**Time consistent measures of risk**

Example: Driving Test Discount

Option 1: Plain Insurance

- Cost: $\$9.00$
- No deductible
- Certain expected outcome:

$$\mathbb{E}[X_1] = -9.00$$

$$\rho(X_1) = \mathbb{E}[X_1] = -9.00$$

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Option 2: Custom Insurance

- Take a safety exam
- Pass with probability $1/2$
  - OK $[\mathbb{P} = 2/3]$: $+\$5.00$
  - Not $[\mathbb{P} = 2/3]$: $-\$20.00$
- Fail with probability $1/2$
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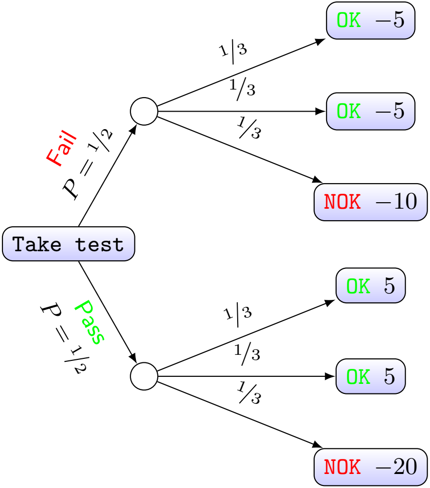
Risk measure: $\rho = \mathrm{CV@R}_{2/3}$

Risk Measure of Option 2

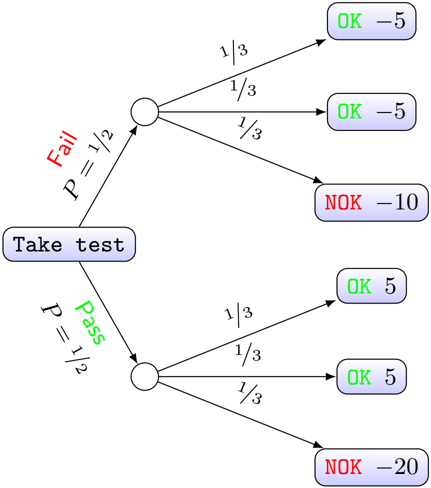


Risk measure:

$$\rho(X_2) = \mathrm{CV@R}_{2/3}(X_2)$$

$\mathbb{P}$	X_2
$1/6$	-5
$1/6$	-5
$1/6$	-10
$1/6$	5
$1/6$	5
$1/6$	-20

Risk Measure of Option 2



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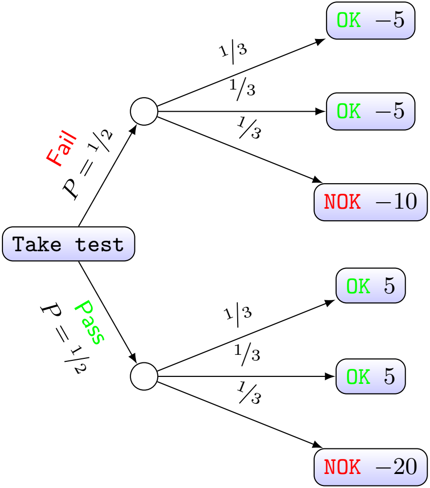
$1/6$	-5
$1/6$	-5
$1/6$	-10
$1/6$	5
$1/6$	5
$1/6$	-20

$$\rho(X_2) = \frac{-5-5-10-20}{4} = $$
$$= -10.0 < -9.0 = \rho(X_1)$$

Risk Measure of Option 2



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$\mathbb{P}$	X_2
$1/6$	-5
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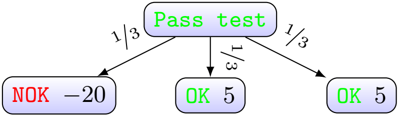
$1/6$	5
$1/6$	5
$1/6$	-20

$$\rho(X_2) < \rho(X_1)$$

Prefer option 1

Optimal Solution of Subproblems

Recall we **prefer option 1**: $\rho(X_1) = -9$



$\mathbb{P}$	$1/3$	$1/3$	$1/3$
X_2	-20	5	5

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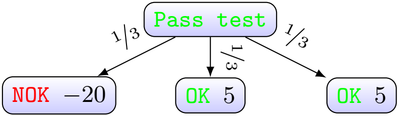


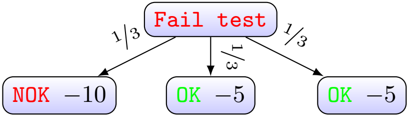
$\mathbb{P}$	1/3	1/3	1/3
X_2	−20	5	5

$$\rho(X_2 \mid \text{Pass}) = \frac{-20 + 5}{2} = -7.5$$

If pass, prefer option 2

Optimal Solution of Subproblems

Recall we **prefer option 1**: $\rho(X_1) = -9$



$\mathbb{P}$	$1/3$	$1/3$	$1/3$
X_2	-10	-5	-5

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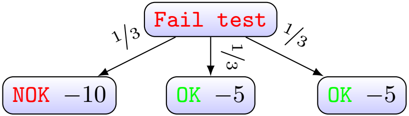

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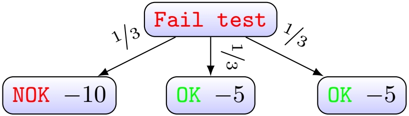

$\mathbb{P}$	$1/3$	$1/3$	$1/3$
X_2	-10	-5	-5

$$\rho(X_2 \mid \text{Fail}) = \frac{-15+5}{2} = -7.5$$

If fail, prefer option 2

Optimal Solution of Subproblems

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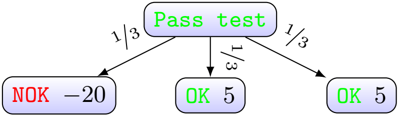

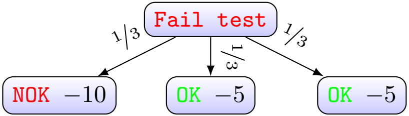

$\mathbb{P}$	$1/3$	$1/3$	$1/3$
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Time inconsistent behavior (Roorda, Schumacher, and Engwerda 2005; Iancu, Petrik, and Subramanian 2015)

Time Consistent Risk Measures

- Filtration (scenario tree) of rewards with T levels:

$$X_1, X_2, X_3, \ldots, X_T$$

- **Dynamic risk measure** at time t:

$$\rho_t(X_t + \cdots + X_T)$$

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- **Time consistent**: if for all X, Y (also dynamic consistent)

$$\rho_{t+1}(X_t + \cdots) \geq \rho_{t+1}(Y_t + \cdots) \Rightarrow \rho_t(X_t + \cdots) \geq \rho_t(Y_t + \cdots)$$

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- Similar to subproblem optimality in programming optimality

Time Consistency via Iterated Risk Mappings

- **Time consistent** risk measures must be composed of **iterated risk mappings** (Roorda, Schumacher, and Engwerda 2005):

$$\mu_1, \mu_2, \ldots, \mu_t$$

- Dynamic risk measure:

$$\rho_t(X_t + \cdots + X_T) = \mu_t(X_t + \mu_{t+1}(X_{t+1} + \mu_{t+2}(x_{t+3} + \cdots)))$$

- Each μ_t: a coherent risk measure applied on subtree of filtration

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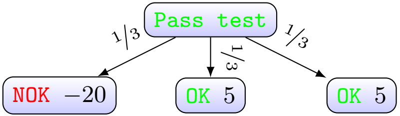
- Each μ_t: a coherent risk measure applied on subtree of filtration
- Markov risk measures for MDPs (Ruszczynski 2010)

Computing Time Consistent Risk Measure



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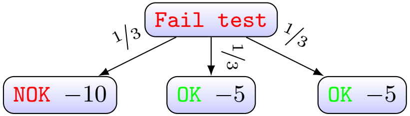
Computing Time Consistent Risk Measure



$$\rho(X_2 \mid \text{Fail}) = \frac{-15 + 5}{2} = -7.5$$

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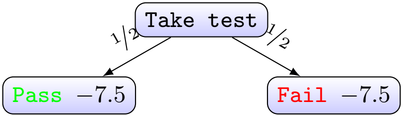
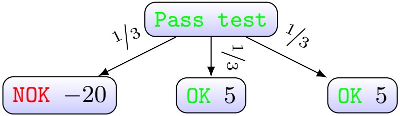
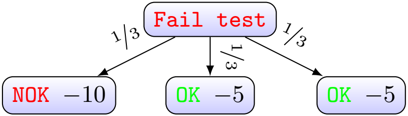


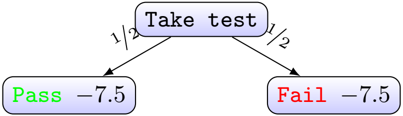
$$\rho(X_2) = \rho(-7.5) = -7.5 > -9$$

Computing Time Consistent Risk Measure





$$\rho(X_2 \mid \text{Pass}) = \frac{-20+5}{2} = -7.5 \qquad \rho(X_2 \mid \text{Fail}) = \frac{-15+5}{2} = -7.5$$



$$\rho(X_2) = \rho(-7.5) = -7.5 > -9$$

Consistently prefer option 1 throughout the execution

Approximating Inconsistent Risk Measures

- Time consistent risk measures are difficult to specify
- Approximate an inconsistent risk measure by a consistent one?
- **Best lower bound**: e.g. what is the best α_1, α_2 such that

$$\mathrm{CV@R}_{\alpha_1}(\mathrm{CV@R}_{\alpha_2}(X)) \le \mathrm{CV@R}_{\alpha}(X) \text{ for all } X$$

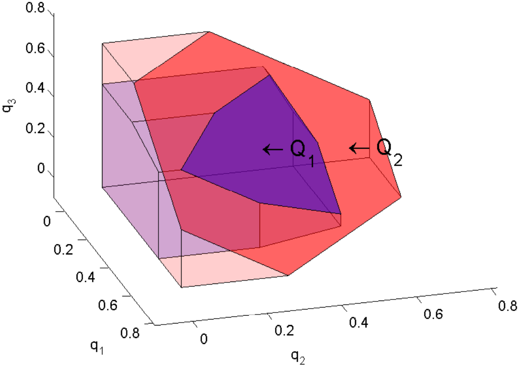
- **Best upper bound**: e.g. what is the best α_1, α_2 such that

$$\mathrm{CV@R}_{\alpha_1}(\mathrm{CV@R}_{\alpha_2}(X)) \ge \mathrm{CV@R}_{\alpha}(X) \text{ for all } X$$

(Iancu, Petrik, and Subramanian 2015)

Best Time Consistent Bounds

- Compare robust sets of consistent and inconsistent measures
- **Main insight**: need to compare *down-monotone* closures of robust sets



Time Consistent Bounds: Main Results

- **Lower consistent bound**:
  - Uniformly tightest bound can be constructed in polynomial time
  - Method: rectangularization
- **Upper consistent bound**:
  - NP hard to even evaluate how tight the approximation is
  - Approximation can be tighter than the lower bound

Planning with Time Consistent Risk Measures

- Stochastic dual dynamic programming (Shapiro 2012)
- Applied in reinforcement learning (Petrik and Subramanian 2012)
- Only entropic dynamically consistent risk measures are law invariant (Kupper and Schachermayer 2006)

Outline

Risk Measures: Many Other Topics

1. Elicitation of risk measures
2. Estimation of risk measure from samples
3. Relationship to acceptance sets
4. Relationship to robust optimization

Take Home Messages

- Coherent risk measures are a convenient and established risk aversion framework

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- Coherent risk measures are a convenient and established risk aversion framework
- Computations with coherent risk measure are more efficient than with utility functions
- Risk measures $(\mathrm{V@R}, \mathrm{CV@R})$ are more intuitive than utility functions
- Time consistency is important in dynamic settings, but can be difficult to achieve (open research problems)
- Risk measures are making inroads in reinforcement learning and artificial intelligence

Thank you!!

Bibliography I

Andreev, Andriy, Antti Kanto, and Pekka Malo (2005). "Closed-Form Calculation of Cvar". In: *Sweedish School of Economics*.

Artzner, Philippe et al. (1999). "Coherent Measures of Risk". In: *Mathematical Finance* 9, pp. 203–228. ISSN: 14679965. DOI: 10.1111/1467-9965.00068.

Ben-Tal, Aharon and Marc Teboulle (2007). "An Old-New Concept of Convex Risk Measures: The Optimized Certainty Equivalent". In: *Mathematical Finance* 17, pp. 449–476. URL: http://onlinelibrary.wiley.com/doi/10.1111/j.1467-9965.2007.00311.x/full.

Bibliography II

Föllmer, Hans and Thomas Knispel (2011). “Entropic Risk Measures: Coherence Vs. Convexity, Model Ambiguity and Robust Large Deviations”. In: *Stochastics and Dynamics* 11.02n03, pp. 333–351. ISSN: 0219-4937. DOI: 10.1142/S0219493711003334. URL: http://www.worldscientific.com/doi/abs/10.1142/S0219493711003334.

Follmer, Hans and Alexander Schied (2011). *Stochastic Finance: An Introduction in Discrete Time*. 3rd. Walter de Gruyter.

Friedman, Daniel et al. (2014). *Risky Curves: On the Empirical Failure of Expected Utility*.

Bibliography III

Iancu, Dan A, Marek Petrik, and Dharmashankar Subramanian (2015). "Tight Approximations of Dynamic Risk Measures". In: *Mathematics of Operations Research* 40.3, pp. 655–682. ISSN: 0364-765X. DOI: 10.1287/moor.2014.0689. arXiv: 1106.6102. URL: http://pubsonline.informs.org/doi/abs/10.1287/moor.2014.0689.

Kupper, Michael and Walter Schachermayer (2006). "Representation results for law invariant time consistent functions". In: *Mathematics and Financial Economics* 16.2, pp. 419–441. URL: http://link.springer.com/article/10.1007/s11579-009-0019-9.

Bibliography IV

- Nadarajah, Saralees, Bo Zhang, and Stephen Chan (2014). "Estimation methods for expected shortfall". In: *Quantitative Finance* 14.2, pp. 271–291. ISSN: 1469-7688. DOI: 10.1080/14697688.2013.816767. URL: http://www.tandfonline.com/doi/abs/10.1080/14697688.2013.816767.
- Petrik, Marek and Dharmashankar Subramanian (2012). "An approximate solution method for large risk-averse Markov decision processes". In: *Uncertainty in Artificial Intelligence (UAI)*. URL: http://arxiv.org/abs/1210.4901.
- Rockafellar, R. Tyrrell and S. Uryasev (2000). "Optimization of conditional value-at-risk". In: *Journal of Risk* 2, pp. 21–41.

Bibliography V

Roorda, Berend, Hans Schumacher, and Jacob Engwerda (2005). "Coherent acceptability measures in multiperiod models". In: *Mathematical Finance*, pp. 1–29. URL: http://onlinelibrary.wiley.com/doi/10.1111/j.1467-9965.2005.00252.x/full.

Ruszczynski, Andrzej (2010). "Risk-averse dynamic programming for Markov decision processes". In: *Mathematical Programming B* 125.2, pp. 235–261. ISSN: 0025-5610. DOI: 10.1007/s10107-010-0393-3. URL: http://link.springer.com/10.1007/s10107-010-0393-3.

Schoemaker, P.J.H. (1980). *Experiments on Decisions under Risk: The Expected Utility Hypothesis*.

Bibliography VI

- Shapiro, A., D. Dentcheva, and A. Ruszczynski (2014). *Lectures on stochastic programming: modeling and theory*. P. 447. ISBN: 089871687X. DOI: http://dx.doi.org/10.1137/1.9780898718751.
- Shapiro, Alexander (2012). "Minimax and risk averse multistage stochastic programming". In: *European Journal of Operational Research* 219.3, pp. 719–726. ISSN: 03772217. DOI: 10.1016/j.ejor.2011.11.005. URL: http://dx.doi.org/10.1016/j.ejor.2011.11.005.
- Shapiro, Alexander, Darinka Dentcheva, and Andrzej Ruszczynski (2009). *Lectures on Stochastic Programming*. SIAM. ISBN: 9780898716870.