http://www.cs.unh.edu/~ruml/cs758
Red-Black Trees

- Searching
- Balanced Trees
- Red-Black Trees
- Rotation
- Insert($z$)
- Fixing Insertion
- Fixup Invariant
- Fix-insert($z$)
- Termination
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Red-Black Trees
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1. AVL Trees (1962)
2. 2-3 Trees
3. red-black trees (1972, popularized 1978)
4. AA trees (1992)
5. left-leaning red-black trees (2008)

probabilistically balanced

1. treaps
2. skip lists
Red-Black Trees

node: data, left, right, parent, color

1. every node is either red or black
2. the root is black
3. (consider nil to be black)
4. both children of a red node are black
5. from any node, all paths to leaves have the same ‘black height’

search and traversal are unchanged
useful subroutines:

- rotate-right
- rotate-left
**Insert(\(z\))**

1. \(z\)'s parent \(\leftarrow\) find-parent(\(z\), root, nil)
2. if parent is nil
3. root \(\leftarrow\) \(z\)
4. else
5. if \(z\) should be before parent
6. parent's left child \(\leftarrow\) \(z\)
7. else
8. parent's right child \(\leftarrow\) \(z\)
9. \(z\)'s children \(\leftarrow\) nil
Insert($z$)

1. $z$’s parent $\leftarrow$ find-parent($z$, root, nil)
2. if parent is nil
3. root $\leftarrow$ $z$
4. else
5. if $z$ should be before parent
6. parent’s left child $\leftarrow$ $z$
7. else
8. parent’s right child $\leftarrow$ $z$
9. $z$’s children $\leftarrow$ nil
10. color $z$ red
11. fix-insert($z$)
Fixing Insertion

Recall properties:

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Cases:

1. red root (property 2)
2. two red in a row (property 4)
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During fixup:

1. \( z \) is red
2. if \( z \)'s parent is the root, it is black
3. at most, property 2 xor 4 is violated at \( z \)
   
   (a) if 2: because \( z \) is root and red
   (b) if 4: because \( z \) and parent are red
Fixup Invariant

Cases:
1. red root (property 2)
2. two red in a row (property 4)

During fixup:
1. $z$ is red
2. if $z$’s parent is the root, it is black
3. at most, property 2 xor 4 is violated at $z$
   (a) if 2: because $z$ is root and red
   (b) if 4: because $z$ and parent are red

Initialization:
1. we colored $z$ red
2. we didn’t touch $z$’s parent, and roots are black
3. just saw this
Fix-insert($z$)

1. while $z$’s parent is red
2. if $z$’s parent is a left child
3. $y \leftarrow z$’s uncle (a right child)
4. if $y$ is red
5. color $z$’s parent black \hspace{1cm} \textit{case 1}
6. color $z$’s uncle $y$ black
7. color $z$’s grandparent red
8. $z \leftarrow z$’s grandparent
9. else if $z$ is a right child
10. $z \leftarrow z$’s parent \hspace{1cm} \textit{case 2}
11. rotate-left($z$)
12. color $z$’s parent black \hspace{1cm} \textit{case 3}
13. color $z$’s grandparent red
14. rotate-right($z$’s grandparent)
15. else, 3 symmetric cases (left↔right)
16. color root black
Assuming other properties are maintained, are we red-black now?

Leverage the invariant:

1. irrelevant
2. irrelevant
3. only 2 xor 4 can be violated in loop
   (a) if 2: root colored black at end, so 2 not violated
   (b) if 4: z’s parent now black, so 4 not violated
Assuming other properties are maintained, are we red-black now?

Leverage the invariant:

1. irrelevant
2. irrelevant
3. only 2 xor 4 can be violated in loop
   (a) if 2: root colored black at end, so 2 not violated
   (b) if 4: $z$’s parent now black, so 4 not violated

How to make progress around loop while maintaining invariant?
central problem: prop 4 violated: $z$ and parent are red

note $z$ has an uncle because the root is black

3 cases (+ 3 more by symmetry of $z$’s parent being left/right):

1. $z$’s uncle $y$ is also red (we have a red layer)
2. $z$’s uncle $y$ is black and $z$ is right child
3. $z$’s uncle $y$ is black and $z$ is left child
central problem: prop 4 violated: \( z \) and parent are red

note \( z \) has an uncle because the root is black

3 cases (+ 3 more by symmetry of \( z \)'s parent being left/right):
1. \( z \)'s uncle \( y \) is also red (we have a red layer)
2. \( z \)'s uncle \( y \) is black and \( z \) is right child
3. \( z \)'s uncle \( y \) is black and \( z \) is left child

Plan:
1. fix case 1, possibly introducing case 2.
2. reduce case 2 to case 3.
3. fix case 3.
Case 1:

Case 1: $z$’s uncle $y$ is also red

Solution: move redness up

1. Color $z$’s parent and uncle black
2. Color grandparent red and recur

Fixup loop invariants:

1. $z$ is red
2. If $z$’s parent is the root, it is black (unchanged)
3. At most, property 2 xor 4 is violated at new $z$. Note previous violations at old $z$ are fixed.

   (a) If 2: because $z$ is root and red
   (b) If 4: because $z$ and parent are red

If new $z$ is root, will be colored black, increasing all heights
Case 2

case 2: $z$’s uncle $y$ is black and $z$ is right child

reduce to case 3: $z$’s uncle $y$ is black and $z$ is left child

rotation doesn’t affect any properties
Case 3

Case 3: $z$’s uncle $y$ is black and $z$ is left child

fix prop 4 at $z$: pull blackness down to $z$’s parent and rotate grandparent under it.

fixup loop invariants:
1. $z$ is red
2. if $z$’s parent is the root, it is black
3. at most, property 2 xor 4 is violated at $z$.
   (a) can’t be prop 2
   (b) if 4: fixed because $z$’s parent is now black
   (c) note black-height is preserved!

We are done and loop will exit
finding place is
finding place is $O(\lg n)$

one fixup iteration is constant time

fixup loops only when moving up, so is
finding place is $O(\lg n)$

one fixup iteration is constant time

fixup loops only when moving up, so is $O(\lg n)$

how many rotations are performed?
For example:

- What’s still confusing?
- What question didn’t you get to ask today?
- What would you like to hear more about?

Please write down your most pressing question about algorithms and put it in the box on your way out.

Thanks!