http://www.cs.unh.edu/~ruml/cs758
Reductions to Graph Problems
Reductions

Graph Problems
- Reductions
  - Clique
  - Vertex Cover
  - Break

Number Problem

CIRCUIT-SAT
  ↓
  SAT
  ↓
  3-CNF SAT
  ↓
  CLIQUE
  ↓
  SUBSET-SUM
  ↓
  VERTEX-COVER
  ↓
  HAM-CYCLE
  ↓
  TSP
Does graph have clique of size \( k \)?

**CLIQUE \( \in \text{NP} \):** given clique, test connectivity (\( k^2 \) time).

**CLIQUE is NP-Hard:** Reduction from 3-CNF SAT! Formula \( \phi \) with \( k \) clauses will be SAT iff graph \( G \) has a \( k \) clique.

For clause \( r \) like \( (l_1^r \lor l_2^r \lor l_3^r) \), add vertices \( v_{i}^r \), \( v_{2}^r \), and \( v_{3}^r \) to \( G \). Add edge from \( v_{i}^r \) to \( v_{s}^j \) iff \( r \neq s \) and \( l_{i}^r \neq \neg l_{j}^s \).

**SAT \( \Rightarrow \) clique:** If \( \phi \) SAT, at least one literal in each clause is true. These form a clique in \( G \) because they cannot conflict.

**Clique \( \Rightarrow \) SAT:** If \( k \) clique, make corresponding literals true. Will satisfy all \( k \) clauses without conflicts.

**Example:** \( (x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_3) \)
Does graph have a vertex cover of size $k$?

VERTEX-COVER $\in$ NP: given cover, check size and that each edge is covered.

VERTEX-COVER is NP-Hard: Reduction from CLIQUE. Form graph complement $\bar{G}$, which has edge $(u, v)$ for $v \neq u$ iff original does not. Claim: $G$ has $k$ clique iff $\bar{G}$ has $|V| - k$ cover.

Cover $\Rightarrow$ clique: All edges in $\bar{E}$ have at least one endpoint in Cover. All pairs $(u, v)$ with both $u$ and $v \notin$ Cover therefore have edge $\in E$. So $V - Cover$ is a clique of size $k$.

Clique $\Rightarrow$ cover: Any edge $(u, v) \in \bar{E}$ implies $\notin E$ implies $u$ or $v$ not in Clique. This implies $u$ or $v$ remains in $V - Clique$ and hence it covers that edge. Size of $V - Clique$ is $|V| - k$. 
Break

Graph Problems
- Reductions
- Clique
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Number Problem

- asst 11
- asst 12
- wildcard topics
Graph Problems

Number Problem
- Reductions
- Subset Sum
- Example Formula
- Subset Sum
- Resulting Set
- EOLQs

Reduction to a Numeric Problem
Reductions

Graph Problems

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  TSP
Given finite set of positive integers, is there a subset that sums to \( t \)?

\[ \text{SUBSET-SUM} \in \text{NP}: \text{given subset, compute sum.} \]

\[ \text{SUBSET-SUM is NP-Hard: Reduction from 3-CNF SAT. Make numbers and the target sum from the formula. For} \ n \ \text{variables and} \ k \ \text{clauses, each number will have} \ n + k \ \text{digits. We ensure no carrying by using base 10 and at most a sum of 6 in each column.} \]

[ see upcoming slide for how to make numbers and target ]

Polynomial time to construct and equivalent to satisfiability.
Example Formula

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\[ C_1 : (x_1 \lor \neg x_2 \lor \neg x_3) \land \]
\[ C_2 : (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land \]
\[ C_3 : (\neg x_1 \lor \neg x_2 \lor x_3) \land \]
\[ C_4 : (x_1 \lor x_2 \lor x_3) \]
Two kinds of numbers:

- Two numbers for each variable, representing positive/negative literals. (These are the ‘important’ ones!) 1 in the variable’s column, and 1 for clauses where that literal appears.
- Clause numbers just allow slop for 1, 2 or 3 true literals per clause.

Target is 1 for each variable and 4 for each clause. Therefore, it requires exactly one form of each variable and at least one true literal in each clause (plus one or both ‘slop numbers’).

Sum $\Rightarrow$ SAT: read off assignment. Target ensures consistency and variable numbers ensure satisfiability.

SAT $\Rightarrow$ sum: construct sum, choosing slop variables last.
### Resulting Set

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For example:

- What’s still confusing?
- What question didn’t you get to ask today?
- What would you like to hear more about?

Please write down your most pressing question about algorithms and put it in the box on your way out.

*Thanks!*