The traffic process linking open, closed, and feedback networks

1 Introduction

The M/M/1 queue with instantaneous Bernoulli feedback, shown in Figure 1, embodies the mystique of product-form networks - the limiting queue length process of a queue with feedback has the same distribution as the ordinary M/M/1 queue (without feedback), even though the two queue systems have completely different traffic processes[1]. External arrivals are drawn from a Poisson distribution with parameter λ. The service time at the queue is drawn from an exponential distribution with rate μ. A job after completing execution instantaneously departs the network with probability q or is fed back to the queue with probability $p = 1-q$. Thus, the input process to the queue consists of new arrivals from outside and returning jobs. The Chapman-Kolmogorov equations (and therefore, the steady state flow balance equations) of this network are identical to those of the ordinary M/M/1 queue with arrival rate λ and service rate $q\mu$. When $\lambda < q\mu$, the limiting queue length distribution for the network exists.

The feedback queue and the ordinary M/M/1 queue are single queue networks. Here, we treat the external Poisson source of arrivals as an "external" queue with infinite number of jobs. The feedback network now consists of two queues - queue 1 is the feedback queue and queue 2 is the external queue as shown in Figure 2. The external queue always has an infinite number of jobs. The state of the network is the number of jobs in the feedback queue.

Observe the state of the network at every transition; this embedded stochastic process is a Markov chain, and the properties of this chain are well known [1, 3]. Each transition in the network occurs as a result of a job completing service at a queue and departing the queue. This departing job then instantaneously arrives at a queue in the network. Thus, every transition is the result of a job departing a queue (external or feedback) and instantaneously arriving at a queue. We observe the state of the network at every transition, but we view the network from the perspective of the job participating in the transition. The job participating in the transition is not included in the state description. Every transition consists of a queue departure and a queue arrival. For every transition, the state of the network can be viewed just after departure and just prior to arrival. We observe the state of the network just prior to every arrival at both queues (feedback and external) in the network. The resulting embedded stochastic process is called the Arrival* process, or in short, the A* process.

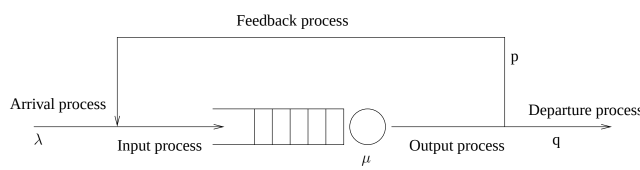

Figure 1: The M/M/1 queue with Bernoulli feedback

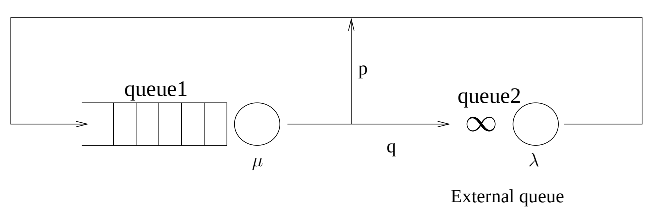

Figure 2: The feedback queue viewed as a network of two queues

[1]The traffic processes refer to the arrival, input, output, feedback, and departure processes.

We show that the A* process is a Markov chain that is recurrent and irreducible. Prior studies of arrival instant traffic processes have viewed the state of the network just prior to arrival at a single queue [2, 4]. The A* traffic process views the network state prior to arrival at every queue in the network. Typically, such traffic processes are very complex [4]. However, this is not the case with the A* traffic process. It is surprising that this traffic process has not been analyzed since it is simple and captures a lot of information about the queueing network.

2 A* Traffic Process

The queue length process for the network is given by:

$$\mathcal{Y} = \{\, Y(\mathsf{T}) : \mathsf{T} \in \mathsf{R}_{+} \}$$

where $Y(\mathsf{T}) = (i, \infty)$ represents the state of the network at time T. There are $i \geq 0$ jobs in the feedback queue and an infinite number of jobs in the external queue. For notational simplicity, the state of the feedback network can be written as:

$$Y(\mathsf{T}) = i; \;\; i \geq 0$$

where i is the number of jobs in the feedback queue at time T. $\mathcal{Y}$ is a Markov process that is irreducible. The state space of $\mathcal{Y}$ is the set $E = \{0, 1, 2, 3, \cdots\}$. If $\lambda < \mathsf{q}\mu$, then $\mathcal{Y}$ has a limiting state distribution [3].

Observe the process $\mathcal{Y}$ just prior to every arrival at the feedback queue and the external queue. Consider the embedded stochastic process

$$\mathsf{X} = \{\mathsf{X}_{\mathsf{n}} = \; Y(\mathsf{T}_{\mathsf{n}}^{-}), \; \mathsf{n} = 1, 2, 3, ..\}$$

where T_n^{-} is time just prior to the n^{th} arrival instant and

$$\mathsf{X}_{\mathsf{n}} = \begin{cases} (i^{*}, \infty) = i^{*} & \text{if the arrival is to feedback queue;} \\ (i, \infty^{*}) = i & \text{if the arrival is to external queue.} \end{cases}$$

The network state is viewed just prior to an arrival, so the arriving job is not included in the state description. The * symbol in the state description denotes the queue at which the job arrives. The embedded queue length process $\mathsf{X} = \{\mathsf{X}_{\mathsf{n}} : \mathsf{n} = 1, 2, \cdots\}$ is called the Arrival* (or A*) process for the network. The state space of X is the set $E \cup E^{*} = \{0, 0^{*}, 1, 1^{*}, 2, 2^{*}, 3, 3^{*}, \cdots\}$.

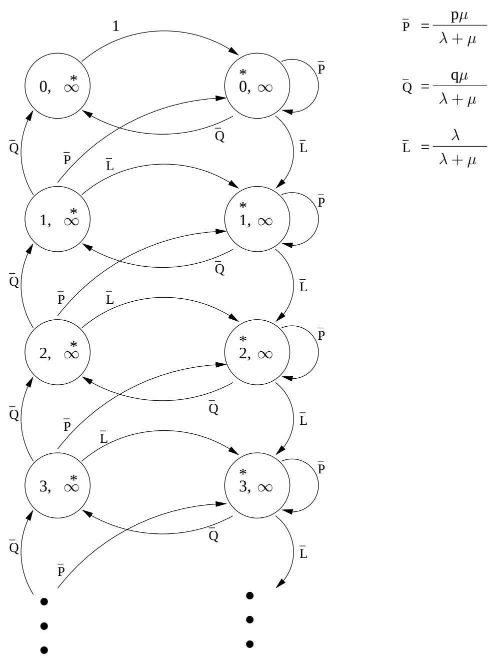

Figure 3: Markov state diagram of the A* process for the feedback network

Theorem 2.1 X *is a Markov chain.*

(The proof is given in a technical paper.) The Markov state diagram of the A* process is given in Figure 3. If $\lambda < \mathsf{q}\mu$, then the system of traffic equations

$$\pi Q = \pi$$

along with the boundary condition $\sum_{E \cup E^{*}} \pi(i) = 1$, has a solution called the stationary distribution [3]. Note that it is possible to get a solution for the stationary equations of the Markov chain even when the Markov chain has no limiting distribution [1, 3]. Let p denote the limiting state probability of the queue length process $\mathcal{Y}$:

$p(i) = \lim_{\mathsf{T}\to\infty} \text{Prob}\{\, Y(\mathsf{T}) = i\}$

The fundamental theorem that relates the stationary distribution of $\mathcal{Y}$ at any instant to the stationary distribution of the embedded Markov chain X [1] leads to:

Theorem 2.2

$$p(i) = \frac{\pi(i-1^*)/\gamma(i-1^*) \;+\; \pi(i)/\gamma(i)}{\sum_{j\in E\cup E^*} \pi(j)/\gamma(j)} \qquad i \in E$$

where $1/\gamma(j)$ *represents the mean sojourn time in state* j*,* $j \in E \cup E^*$.

Result 2.1 *The system of traffic equations of the A* process reduce to:*

$$\begin{aligned} \lambda\pi(i) &= \mathsf{q}\mu\pi(i+1) \quad i = 0,1,2,\cdots \\ \lambda\pi(i^*) &= \mathsf{q}\mu\pi(i+1^*) \quad i^* = 0^*,1^*,2^*,\cdots \end{aligned}$$

These are identical to the local balance equations for both the feedback queue and the ordinary M/M/1 queue with parameters λ *and* $\mathsf{q}\mu$*.*

Result 2.2

$$\mathsf{V}(2)\sum_{i=0}^{\infty}\pi(i^*) = \mathsf{V}(1)\sum_{i=0}^{\infty}\pi(i)$$

where $\mathsf{V}(1)$ *and* $\mathsf{V}(2)$ *are the relative visit counts for the network.*

Result 2.3

$$\mathsf{V}(2)\pi(i^*) = \mathsf{V}(1)\pi(i)$$

Let $\mathsf{X}^{\mathsf{a}}_{\mathsf{n}}$ represent the state of the Markov process $\mathcal{Y}$ just prior to job arrivals at the feedback queue. The job arrivals include external arrivals and feedback arrivals. The A* process incorporates both the arrival instant queue information and the visit count information. The embedded Markov process $\mathsf{X}^{\mathsf{a}}_{\mathsf{n}}$ only incorporates the arrival instant queue information. The following result follows:

Result 2.4 *Let* $p^a(i)$ *represent the limiting probability of finding* i *jobs in the feedback queue just prior to arrival at the feedback queue.*

$$p^a(i) = C^a \times \pi(i^*), \quad i = 0,1,2,\cdots$$

where $C^a = \frac{1}{\sum_{i=0}^{\infty}\pi(i^*)}$.

The A* process of the M/M/1 queue can be similarly analyzed.

3 Conclusion

This paper establishes the A* traffic process that is generated by observing the state of a network just prior to every arrival instant at all the queues in the network. In addition to a queue's arrival instant distribution, this traffic process captures a lot of information about the network, such as visit counts, network configuration, and service rates. The underlying Markov chain could be periodic or non-periodic depending on the configuration. The A* traffic process is used to explain why the feedback queue has identical queue length distribution as an ordinary M/M/1 queue. Most of the well known results on product-form networks, such as the Arrival theorem, can be proved using the A* process. The A* process could be used to understand the behavior of non-product form networks.

References

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