# The M/M/1 Fork-Join Queue with Variable Sub-Tasks

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### Abstract

The fork-join queue models parallel resources where arriving jobs divide into various number of sub-tasks that are assigned to unique devices within the parallel resource. Each device in the parallel resource is modeled by M/M/1 queueing servers. A job completes execution and departs the parallel resource after all its sub-tasks complete execution. This paper analyzes N-server fork-join queues where arriving jobs divide into  $1 \le K \le N$  sub-tasks that are assigned to unique servers of the fork-join queue. There is no known closed-form solution for N > 2 fork-join queues. The paper presents an O(log K) algorithm for computing the mean response time pessimistic and optimistic bounds and for computing the mean response time approximation of the fork-join queue. The error bounds for the response time bounds and approximation are presented.

Index Terms: fork-join synchronization, performance evaluation, parallel computer and storage systems.

## **1** Introduction

Modern computer systems rely on parallel resources, such as multiple processors and disk arrays, to satisfy the performance requirements of its application programs. For example, the response time of an application program is reduced by concurrently executing sub-parts of the program on multiple processors. Similarly, the I/O throughput of a storage system is increased by accessing data from multiple disks. The jobs submitted to a parallel resource are divided into sub-tasks that are each submitted to separate devices within the parallel resource. A job completes execution and departs the parallel resource only after all its sub-tasks complete.

From a performance analysis viewpoint, the N devices of a parallel resource are modeled by N parallel queueing servers jointly referred to as a *fork-join* queue. Figure 1 presents a fork-join queue. Jobs arrive at the fork-join queue at rate  $\lambda$ . Upon arrival, each job divides (at the fork point) into K identical sub-tasks, where  $1 \le K \le N$ . Based on a pre-defined allocation policy, each of these K sub-tasks is submitted to a unique server within the fork-join queue. The probability that a particular server is assigned a sub-task of an arriving job is given by server\_access\_probability. The queueing discipline is first-come-first-served. The sub-tasks at each server are serviced at rate  $\mu$ . When a sub-task completes execution, it will wait (at the join point) until all its sibling sub-tasks complete execution. A job completes execution and departs the fork-join queue after all its sub-tasks complete. In this paper, we analyze fork-join queues with exponential inter-arrival and service times. That is, the N servers of the fork-join model are M/M/1 queues with synchronized arrivals.

Due to the wide-spread use of parallelism in computer and storage systems, the fork-join queue has been studied extensively. Section 2 summarizes the fork-join literature. An exact analysis of the fork-join queue is presented only for 2-server fork-join queues [2, 7, 18, 23]. There is no known closed-form solution for N > 2 server fork-join queues. Hence, the performance measures of fork-join queues with N > 2 servers is computed using approximation and bounding techniques. This paper also presents mean performance bounds and approximations for the fork-join queue. The **contribution** of this paper is that it is the first to analyze the M/M/1 N-server fork-join queue where jobs divide into  $1 \le K \le N$  sub-tasks. All previous papers on M/M/1 fork-join queues assume that every job divides into N sub-tasks. It is important to analyze fork-join queues where jobs divide into K < N sub-tasks because such fork-join queues model the behavior of real parallel systems. For example, a parallel job may only be executed on some of the

#### Sub-task service area



Figure 1: Fork-join queue

processors of a multi-processor system. Similarly, the I/O requests submitted to a disk-array may only access some of the disks in the array.

This paper presents simple pessimistic and optimistic mean response time bounds and a mean response time approximation for the fork-join queue with  $K \leq N$  sub-tasks. The complexity of the response time bounds and approximation computation is O(log K). This paper also presents the error bound for the response time approximation. Comparison of the mean response time approximations against simulations show an average error of 5% for K (*i.e.*, degree of parallelism) varying from 2 to 100 and parallel resource utilization varying from 0.1 to 0.9.

The remainder of this paper is organized as follows. Section 2 summarizes related work on performance analysis of fork-join queues. Section 3 presents a Markov analysis of the M/M/1 fork-join queue. Sections 4 and 5 use this Markov analysis to derive the mean response time bounds and response time approximation of fork-join queues. Finally, conclusions and future work are presented in Section 6.

# 2 Related Work

Several papers study parallel (fork-join) queues and propose tools for analyzing their performance. Exact performance measures have been derived only for fork-join queues with two servers [2, 7, 18, 23]. Of these, [2] and [7] derive exact steady state distribution for two server fork-join queues in open networks, and [18] and [23], respectively, derive exact mean response times of two server fork-join queues in open and closed networks. Results in [2] assumes general service time distribution, and results in [7], [18], and [23] assume exponential service time distributions. Due to the difficulty of analyzing fork-join queues exactly, all studies on fork-join queues with three or more servers are approximation or bounding analysis. Heidelberger and Trivedi [8] consider a closed queuing network in which jobs divide into two or more asynchronous tasks. The join synchronization is not modeled. The service centers are of a type described in the BCMP theorem. They develop an iterative method for solving a sequence of product-form models. In [9], the model is expanded to include the join synchronization. Nelson and Tantawi [18] consider a scaling approximation technique to analyze the mean response time of an open homogeneous fork-join queue with exponential service time distributions. They assume that the mean response time increases at the same rate as the increase in the degree of parallelism. Closed-form approximation expressions for the mean response time are developed. An extension of this approximation to heavy traffic, relying on a light traffic interpolation technique, is developed by Makowski and Varma [17]. Kim and Agrawala [10] analyze waiting times for two server open, homogeneous fork-join queues with exponential and 2-stage Erlang service time distributions. In [15, 16], Lui, Muntz, and Towsley present a bounding technique for an open, homogeneous fork-join network with a k-stage Erlang distribution. Response time bounds are obtained for acyclic fork-join queuing networks by Baccelli et. al. [3] using stochastic ordering principles

and association of random variables. In [4], Baccelli and Liu propose a new class of queuing models for evaluating the performance of parallel systems. Using the concept of associated random variables, Kumar and Shorey [11] obtain response time bounds for an open fork-join model in which a job forks into a random number of tasks. Service times are drawn from a general distribution. Balsamo, Donatiello, and Van Dijk [5] propose a matrix-geometric algorithmic approach for computing performance bounds of open heterogeneous fork-join systems. Ray [6] uses a dynamic-bubblesort analysis technique to develop a response time bound for fork-join queues with exponential service times. Varki [23] presents a response time approximation for fork-join queues that generalizes the response time expression for single server queues.

There are fewer papers on fork-join queues in closed networks. Almeida and Dowdy [1] propose an iterative technique for obtaining lower performance bounds of closed fork-join networks with exponential service times. No proofs for the technique are presented. Liu and Perros [13, 14] propose an approximation procedure based on decomposition and aggregation for analyzing a closed queuing system with K-sibling fork-join queues. Their method provides an upper bound for mean response time. In [22], Varki develops a mean-value analysis technique for closed fork-join parallel networks. The fork-join structure is studied with relation to parallel storage systems (RAID) in [12, 19, 20].

All but one of the papers on open fork-join queues assume that arriving jobs split into exactly N sub-tasks upon arrival at a N server fork-join queue. In [11], mean performance bounds are computed for M/G/1 fork-join queues where jobs divide into random number of sub-tasks. This is the first paper to analyze M/M/1 fork-join queues where jobs divide into K  $\leq N$  sub-tasks. It is important to compute performance measures of such fork-join queues since these queues model the behavior of parallel computer resources such as disk-arrays and multi-processor computers. Furthermore, the response time computations presented here are computationally simple and scale well with increasing parallelism and increasing load.

### **3** Markov Analysis

We use Markov state diagrams to analyze the fork-join queue. This analysis is then used in Sections 4 and 5 to derive response time bounds and approximations. Let  $P_N$  represent a N-server fork-join queue where every arriving job divides into N sub-tasks that are assigned to the N servers. The state of  $P_N$  is represented by the vector  $(n_1, \dots, n_N)$ , where  $n_i$  represents the number of sub-tasks at a server of  $P_N$ . Since the N service centers are identical, the  $n_i$ 's are ordered such that  $n_1 \le n_2 \le \dots \le n_N$ . Thus,  $n_N$  is equal to the number of tasks at the longest server queue. The server queueing discipline is first-come-first-served and each job divides into N sub-tasks, one for each server. Hence,  $n_N$  also represents the total number of jobs in  $P_N$ .

Figure 2 presents the Markov diagram of  $P_2$ . The diagram maps the states of  $P_2$  when there are 0, 1, 2, and 3 jobs in  $P_2$ . Column *i* (*i* = 0, 1, 2, 3) of the diagram represents states with *i* jobs in the fork-join queue. Row *i* (*i* = 0, 1, 2, 3) of the diagram represents states with *i* sub-tasks at the join point. The horizontal transition arcs represent the arrival of jobs at  $P_2$ . The downward transition arcs,  $\vec{t_1}$ , represent the movement of a sub-task to the join point. The diagonal transition arcs,  $\vec{t_2}$ , represent the movement of the last sub-task of a job to the join point at which instant this job departs  $P_2$ . The time spent by a job in  $P_2$  can be factored into two phases, namely, phase<sub>2</sub> and phase<sub>1</sub>, in order. In phase<sub>2</sub>, two sub-tasks of the job are waiting for, or receiving, service at the service centers of  $P_2$ . In phase<sub>1</sub>, only one sub-task of the job is at the service center while its sibling sub-task waits at the join point.

Next, we analyze the Markov state diagram of  $P_3$  given in Figure 3. The horizontal transition arcs again represent the movement of jobs into  $P_3$  at rate  $\lambda$ . The arcs  $\vec{t_k}$  (k = 1, 2, 3) represent the movement of the  $k^{th}$  sub-task of a job to the join point. The response time of a job in  $P_3$  can be factored into three phases. In general, the response time of a job in  $P_N$  can be factored into N phases, namely,  $phase_N, \dots, phase_1$ , in order. A phase,  $phase_k$ , represents the situation when k sub-tasks of the job are at the service centers. A phase,  $phase_k$ , ends with the movement of one of



Figure 2: Markov diagram of P<sub>2</sub>

the executing sub-tasks to the join point, at which point the corresponding job moves to  $phase_{k-1}$  of its response time.

The time spent completing each phase of a job's response time in P<sub>N</sub> can be viewed as the time spent getting service at N non-parallel queueing centers Serial<sub>N</sub>, Serial<sub>N-1</sub>, ..., Serial<sub>1</sub>, in order [23]. A job at service center Serial<sub>k</sub> is in *phase<sub>k</sub>* of its response time. Let Serial\_P<sub>N</sub>\_model represent the Serial<sub>N</sub>, Serial<sub>N-1</sub>, ..., Serial<sub>1</sub> model of P<sub>N</sub>. In the Serial\_P<sub>N</sub>\_model, let  $n_{\text{Serial}_k}$  represents the number of jobs in the Serial<sub>k</sub> queue. By construction,  $n_{\text{Serial}_k}$  represents the number of jobs in the Serial<sub>k</sub>  $n_{\text{Serial}_{N-1}}$ ; ...;  $n_{\text{Serial}_2}$ ;  $n_{\text{Serial}_1}$ ) of Serial\_P<sub>N</sub>\_model is equivalent to the state ( $n_{\text{Serial}_N}$ ,  $n_{\text{Serial}_{N-1}}$ ,  $n_{\text{Serial}_{N-1}}$ ,  $n_{\text{Serial}_{N-1}}$ ,  $n_{\text{Serial}_2}$ ,  $n_{\text{Serial}_1}$ ,  $n_{\text{Serial}_N}$ ,  $n_{\text{Serial}_$ 

**Example 1** The state (3,3,3) of P<sub>3</sub> represents the state when all sub-tasks of the 3 jobs within P<sub>3</sub> are at the servers. Thus, all 3 jobs are in the first phase (phase<sub>3</sub>) of their response time, which implies that all 3 jobs are at Serial<sub>3</sub>. This state is equal to the state (3;0;0) of the serial model.

The state (0,3,4) of P<sub>3</sub> represents the state with job-1 in phase<sub>1</sub> of its response time with 1 active sub-task; job-2, job-3, and job-4 are in phase<sub>2</sub> of their response time with 2 active sub-tasks each. Thus, job-1 is at server Serial<sub>1</sub>, job-2, job-3, and job-4 are at server Serial<sub>2</sub>. This is equivalent to state (0;3;1) of the serial model.

There is a 1-1 and onto mapping from the state space of the Serial\_P<sub>N</sub>\_model to the state space of P<sub>N</sub>. (That is, every state in P<sub>N</sub> can be mapped to a state in Serial\_P<sub>N</sub>\_model, and vice-versa.) Set the rates along the transition arcs in the Markov diagram of Serial\_P<sub>N</sub>\_model to be equal to the rates along the corresponding transition arcs of P<sub>N</sub>. Figure 4 presents the Markov diagram of the Serial\_P<sub>N</sub>\_model. By construction, P<sub>N</sub> and Serial\_P<sub>N</sub>\_model have identical Markov processes and are equivalent models.

The advantage of the serial model is that the fork-join queue,  $P_N$ , can be analyzed from the viewpoint of a job's response time at the parallel queue. If a job arriving at  $P_N$  divides into  $1 \le K \le N$  sub-tasks, then the response time of a job can be mapped to the response time of a job at the Serial\_ $P_K$ \_model, the serial model equivalent to  $P_K$ . In the next section, we use this serial mapping of the fork-join queue to compute mean performance bounds and approximate performance measures of the fork-join queues.



Figure 3: Markov diagram of  $P_3$ 



Figure 4: Markov diagram of Serial\_ $P_N$ \_model

## 4 N Sub-Tasks

This section presents the derivation of the response time bounds and the response time approximation for the N-server fork-join queue where every job divides into exactly N sub-tasks. Before presenting this derivation, we briefly explain the harmonic number and the partial sum of a sequence since both are used in the remainder of this paper.

### 4.1 Harmonic Number and Partial Sums

A well known result in probability theory is that when there are K identical sub-tasks executing concurrently on exponential servers, the mean time taken to finish executing the K sub-tasks is  $H_K/\mu$ , the mean of the K<sup>th</sup> order statistic of sub-task execution times [21]. Here,  $1/\mu$  represents the mean execution time of a sub-task, and

$$\mathsf{H}_{\mathsf{K}} = 1 + \frac{1}{2} + \dots + \frac{1}{\mathsf{K}}$$

represents the  $K^{th}$  harmonic number. Since the response time of a job at a fork-join queue is the time taken from arrival instant until all the K sub-tasks of the job complete execution, the harmonic number plays a key role in the response time computation of fork-join queues.

We now define the K<sup>th</sup> partial sum, Sum<sub>a<sub>K</sub></sub>, of a sequence a. Consider a sequence  $a = \left\langle \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots, \frac{1}{a_K}, \cdots \right\rangle$ . The K<sup>th</sup> partial sum, Sum<sub>a<sub>K</sub></sub> of the sequence is given by:

$$Sum_{a_K} = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_K}$$

The K<sup>th</sup> harmonic number is the K<sup>th</sup> partial sum, Sum<sub>K</sub>, of the sequence  $a = < 1, 1/2, 1/3, \dots, 1/K, \dots >$  where  $a_{K} = K$ . That is,

$$H_{K} = Sum_{K}$$

### 4.2 **Response Time Bounds and Approximation**

We first present the mean response time pessimistic and optimistic bounds and then use these bounds to compute the response approximation. The parameter  $\rho = \lambda/\mu$  represent the utilization of a server within the fork-join queue. Let  $R_N$  represent the mean response time of a N-server fork-join queue. The next theorem presents optimistic and pessimistic bounds of  $R_N$ .

**Theorem 4.1** The mean response time,  $R_N$ , of a N-server fork-join queue where each arriving job divides into N sub-tasks is bounded by

$$\frac{1}{\mu} \left( \mathsf{H}_{\mathsf{N}} + \rho * \mathsf{Sum}_{\mathsf{N}(\mathsf{N}-\rho)} \right) \leq \mathsf{R}_{\mathsf{N}} \leq \frac{\mathsf{H}_{\mathsf{N}}}{\mu} \left( 1 + \frac{\rho}{1-\rho} \right)$$

where  $\rho = \lambda/\mu$  is the utilization of a server in the fork-join queue,

 $\begin{aligned} \mathsf{H}_{\mathsf{N}} &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{\mathsf{N}} \text{ is the } \mathsf{N}^{th} \text{ harmonic number, and} \\ \mathsf{Sum}_{\mathsf{N}(\mathsf{N}-\rho)} &= \frac{1}{1-\rho} + \frac{1}{2} \frac{1}{2-\rho} + \frac{1}{3} \frac{1}{3-\rho} + \dots + \frac{1}{\mathsf{N}} \frac{1}{\mathsf{N}-\rho} \text{ is the } \mathsf{N}^{th} \text{ partial sum of the sequence } \left\langle \frac{1}{1-\rho}, \frac{1}{2} \frac{1}{2-\rho}, \dots, \frac{1}{\mathsf{N}} \frac{1}{\mathsf{N}-\rho}, \dots \right\rangle \end{aligned}$ 

**Proof:** The pessimistic bound is proved in [18] using associated random variables. Here, we present an informal argument for the pessimistic bound.

Let R<sub>1</sub> represent the mean service time of a single M/M/1 queue. Then,  $H_N * R_1$  represents the response time of the M/M/1 fork-join queue if the service time of each job at the parallel queue equals  $H_N/\mu$ . The service time of a parallel job equals  $H_N/\mu$  only if all the N sub-tasks of the job execute concurrently. In a parallel fork-join queue, however, some sub-tasks of a job may have completed execution and be at the join-point, while other sub-tasks of the job are waiting for service or receiving service. Thus, due to the presence of N independent queues, the sub-tasks of a job in a fork-join queue may not all execute at the same time. This gives

$$\begin{array}{rcl} \mathsf{R}_{\mathsf{N}} & \leq & \mathsf{H}_{\mathsf{N}} \ast \mathsf{R}_{1} \\ & = & \displaystyle \frac{\mathsf{H}_{\mathsf{N}}}{\mu} \left( 1 + \displaystyle \frac{\rho}{1-\rho} \right) \end{array}$$

The optimistic bound is proved using the Markov analysis presented in the previous section. By construction, the response time of a job in  $P_N$  is equal to the response time of the job in the Serial\_ $P_N$ \_model. The response time of a job in the Serial\_ $P_N$ \_model is equal to the sum of the response times at the Serial\_N, Serial\_ $N_-1$ ,  $\cdots$ , Serial\_1 queues. The mean service rate at Serial\_k can equal  $\mu$ ,  $2\mu$ ,  $3\mu$ ,  $\cdots$ ,  $(k-1)\mu$ , or  $k\mu$  depending on the number of jobs at queues Serial\_ $k_{-1}$ ,  $\cdots$ , Serial\_2, and Serial\_1. (Refer to Appendix A for details on the service rate at a server in the Serial\_ $P_N$ \_model.) Thus, the overall mean service rate at Serial\_k (k > 1) is less than  $k\mu$ . Let  $R_1(k\mu)$  represent the mean response time of a M/M/1 queue with arrival rate  $\lambda$  and service rate  $k\mu$ . Since the mean service rate at the Serial\_k queue lies between  $[\mu, k\mu]$ , the mean response time at Serial\_k (k > 1) is greater than the response time  $R_1(k\mu)$ . That is,

$$R\_Serial_k \ge R_1(k\mu)$$

This gives,

$$\begin{aligned} \mathsf{R}_{\mathsf{N}} &\geq & \mathsf{R}_{1}(\mu) + \mathsf{R}_{1}(2\mu) + \mathsf{R}_{1}(3\mu) + \dots + \mathsf{R}_{1}(N\mu) \\ &= & \frac{1}{\mu} \left( \mathsf{H}_{\mathsf{N}} + \frac{\rho}{1-\rho} + \frac{1}{2} \frac{\rho}{2-\rho} + \frac{1}{3} \frac{\rho}{3-\rho} + \dots + \frac{1}{N} \frac{\rho}{N-\rho} \right) \\ &= & \frac{1}{\mu} \left( \mathsf{H}_{\mathsf{N}} + \rho * \mathsf{Sum}_{\mathsf{N}(\mathsf{N}-\rho)} \right) \end{aligned}$$

Let  $R_{opt}$  and  $R_{pes}$ , respectively, represent the optimistic and pessimistic response time bounds computed in Theorem 4.1. That is,

$$\mathsf{R}_{\mathsf{opt}} = \frac{1}{\mu} \left( \mathsf{H}_{\mathsf{N}} + \rho * \mathsf{Sum}_{\mathsf{N}(\mathsf{N}-\rho)} \right)$$

$$\mathsf{R}_{\mathsf{pes}} = \frac{\mathsf{H}_{\mathsf{N}}}{\mu} \left( 1 + \frac{\rho}{1-\rho} \right) = \frac{1}{\mu} \left( \mathsf{H}_{\mathsf{N}} + \rho * \mathsf{Sum}_{\mathsf{N}(1-\rho)} \right)$$

The difference between Rpes and Ropt is

$$\begin{aligned} \mathsf{R}_{\mathsf{diff}} &= \mathsf{R}_{\mathsf{pes}} - \mathsf{R}_{\mathsf{opt}} \\ &= \frac{\rho}{\mu} (\mathsf{Sum}_{\mathsf{N}(1-\rho)} - \mathsf{Sum}_{\mathsf{N}(\mathsf{N}-\rho)}) \\ &= \frac{1}{\mu} \frac{\rho}{1-\rho} \left( \frac{1}{2(2-\rho)} + \frac{2}{3(3-\rho)} + \frac{3}{4(4-\rho)} + \dots + \frac{\mathsf{N}-1}{\mathsf{N}(\mathsf{N}-\rho)} \right) \end{aligned}$$

The maximum error in the spread of the response time bounds is given by

Maximum error = 
$$\frac{\mathsf{R}_{\mathsf{diff}}}{\mathsf{R}_{\mathsf{pes}} + \mathsf{R}_{\mathsf{opt}}} * 100$$

For a given  $\rho$ , the value of  $R_{diff}$  is non-decreasing for increasing values of N. For a given N, the value of  $R_{diff}$  is increasing for increasing  $\rho$ . The bounds are tight for  $\rho \le 0.6$ . If  $\lambda = 1$  time unit,  $R_{diff} = 0.6$  time unit when  $\rho = 0.1$  and N = 1000;  $R_{diff} = 3.05$  time units when  $\rho = 0.5$  and N = 1000;  $R_{diff} = 5.5$  time units when  $\rho = 0.6$  and N = 1000. The relative difference between the bounds increases for  $\rho > 0.6$ .  $R_{diff} = 10.2$  time units when  $\rho = 0.7$  and N = 1000;  $R_{diff} = 20.2$  time units when  $\rho = 0.8$  and N = 1000;  $R_{diff} = 33.2$  time units when  $\rho = 0.9$  and N = 1000;  $R_{diff} = 51.8$  time units when  $\rho = 0.9$  and N = 1000. To address this spread in the bounds for  $\rho > 0.6$ , we present a response time approximation that is relatively invariant to the values of  $\rho$  and N. The next corollary presents the response time approximation, computed from the response time optimistic and pessimistic bounds.

### **Corollary 4.1**

$$\begin{aligned} \mathsf{R}_{\mathsf{N}} &\approx \quad \frac{\mathsf{R}_{\mathsf{opt}} + \mathsf{R}_{\mathsf{pes}}}{2} \\ &= \quad \frac{1}{\mu} \left( \mathsf{H}_{\mathsf{N}} + \frac{\rho}{2(1-\rho)} \left( \mathsf{Sum}_{\mathsf{N}-\rho} + (1-2\rho) * \mathsf{Sum}_{\mathsf{N}(\mathsf{N}-\rho)} \right) \right) \end{aligned}$$

where  $\rho = \frac{\lambda}{\mu}$ ,  $H_N = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N}$ ,  $Sum_{N-\rho} = \frac{1}{1-\rho} + \frac{1}{2-\rho} + \frac{1}{3-\rho} + \dots + \frac{1}{N-\rho}$ ,  $Sum_{N(N-\rho)} = \frac{1}{1-\rho} + \frac{1}{2}\frac{1}{2-\rho} + \frac{1}{3}\frac{1}{3-\rho} + \dots + \frac{1}{N}\frac{1}{N-\rho}$ .

Let  $R_{approx}$  represent the mean response time approximation computed in Corollary 4.1. The tightness of the response time approximation is verified by comparing against simulation results for N varying from 2, 3, 4, ..., 99, 100 and  $\rho$  varying from 0.1 to 0.9. The simulated response time,  $R_{simulation}$ , is accurate within 1% at 95 percent confidence. Figure 5 plots the mean simulated response times, the mean response time bounds, and the mean approximate response times for N-server fork-join queues where every job divides into N sub-tasks. There are three graphs in the figure corresponding to  $\rho$  values of 0.1, 0.5, and 0.9. The error in the response time approximation is given by

Relative approximation error 
$$= \frac{R_{approx} - R_{simulation}}{R_{simulation}} * 100$$

Tables 1 and 2 present the approximate and simulated mean response time values and the error in the approximation. The errors increase for increasing values of  $\rho$  and are maximum ( $\approx 13\%$ ) for  $\rho = 0.9$ . An interesting point is that the errors are relatively invariant to the value of N. That is, for a fixed  $\rho$ , the relative error remains approximately constant as N increases. On analyzing the response times for  $\rho = 0.1, 0.2, 0.3, \dots, 0.9$ , we observe that the approximate response time is a pessimistic bound for  $\rho < 0.5$  and becomes an optimistic bound for  $\rho \ge 0.5$ . This suggests that at  $\rho \approx = 0.5$  the approximate response time matches the exact response time, though mathematical analysis is required to verify this observation.



Figure 5: Model and simulated mean response time values when K = N

# 5 $1 \leq K \leq N$ Sub-Tasks

This section presents the response time optimistic and pessimistic bounds and the response time approximation for N-server fork-join queues where arriving jobs divide into  $1 \le K \le N$  sub-tasks. Let  $P_N^K$  represent such a fork-join queue. The workload is single class so all jobs divide into K sub-tasks that each have a mean service time of  $1/\mu$ . The arrival rate to  $P_N^K$  is  $\lambda$ . The arrival rate to servers of  $P_N^K$ , however, is less than  $\lambda$ . The server arrival rate can be computed from the value of K and the sub-task allocation policy. For example, if the K sub-tasks of a job can only be submitted to K adjoining service centers of the fork-join queue, then the arrival rate to service centers is given by  $K/N * \lambda$ . Let server\_access\_probability represent the probability that an arriving job's sub-tasks are submitted to a server. In the above example, server\_access\_probability = K/N. If all arriving jobs divide into N sub-tasks then server\_access\_probability = 1. In general, if we assume a allocation policy that treats each server uniformly, then server\_access\_probability = K/N. The next theorem shows that the response time of  $P_N^K$  with arrival rate  $\lambda$  is equal to the response time of  $P_K$  with arrival rate  $\lambda *$  server\_access\_probability.

**Theorem 5.1** The response time,  $R_N^K$ , of a N-server fork-join queue  $P_N^K$  with arrival rate  $\lambda$  and service rate  $\mu$  and where each arriving job divides into  $1 \le K \le N$  sub-tasks is equal to the response time,  $R_K$ , of a K-server fork-join queue  $P_K$  with arrival rate  $\lambda * K/N$  and service rate  $\mu$  and where each arriving job divides into K sub-tasks.

**Proof:** Consider  $P_N^K$  (a N-server fork-join queue where each arriving job divides into  $1 \le K \le N$  sub-tasks) with arrival rate lambda and service rate mu. The probability that a server queue in  $P_N^K$  is assigned a sub-task is K/N. Therefore, the arrival rate to a server queue of  $P_N^K$  is  $\lambda * K/N$ . That is, each server of  $P_N^K$  is a M/M/1 queue with arrival rate  $\lambda * K/N$  and service rate  $\mu$ .

Now, consider  $P_K$  (a K-server fork-join queue where each arriving job divides into K sub-tasks) with arrival rate  $\lambda * K/N$  and service rate  $\mu$ . Thus, each server of  $P_K$  is a M/M/1 queue with arrival  $\lambda * K/N$  and service rate  $\mu$ .

Hence, a M/M/1 queue of the  $P_N^K$  fork-join queue is identical to a M/M/1 queue of the  $P_K$  fork-join queue.

In both fork-join queues  $P_N^K$  and  $P_K$ , the response time of a job is the time taken for all K sub-tasks of the job to finish. The response time of a job at the  $P_N^K$  fork-join queue is only dependent on the state of the K queues it is assigned to. To a job arriving at the  $P_N^K$  fork-join queue, the K M/M/1 queues assigned to the job's sub-tasks are statistically identical to the K M/M/1 queues of the  $P_K$  fork-join queue. Hence, the response time of a job at the  $P_N^K$  fork-join queue is identical to the response time of a job at the  $P_K$  fork-join queue.

An implication of Theorem 5.1 is that by setting  $\rho = (\lambda * \text{server\_access\_probability})/\mu$ , and N = K, Theorem 4.1 and Corollary 4.1, respectively, can be used to compute the response time bounds and the response time approximation of the P<sup>K</sup><sub>N</sub> fork-join queue.

**Corollary 5.1** The mean response time,  $R_N^K$ , of a N-server fork-join queue,  $P_N^K$ , where each arriving job divides into  $1 \le K \le N$  sub-tasks is bounded by

$$\frac{1}{\mu} \left( \mathsf{H}_{\mathsf{K}} + \rho * \mathsf{Sum}_{\mathsf{K}(\mathsf{K}-\rho)} \right) \leq \mathsf{R}_{\mathsf{N}}^{\mathsf{K}} \leq \frac{\mathsf{H}_{\mathsf{K}}}{\mu} \left( 1 + \frac{\rho}{1-\rho} \right)$$

The mean response time  $\mathsf{R}_N^K$  of  $\mathsf{P}_N^K$  is approximated by

$$\mathsf{R}_{\mathsf{N}}^{\mathsf{K}} \approx \frac{1}{\mu} \left( \mathsf{H}_{\mathsf{K}} + \frac{\rho}{2(1-\rho)} \left( \mathsf{Sum}_{\mathsf{K}-\rho} + (1-2\rho) * \mathsf{Sum}_{\mathsf{K}(\mathsf{K}-\rho)} \right) \right)$$

where  $\rho = \frac{\lambda * K}{\mu * N}$  is the utilization of a server in the fork-join queue,  $H_{K} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{K}$  is the K<sup>th</sup> harmonic number,  $Sum_{K(K-\rho)} = \frac{1}{1-\rho} + \frac{1}{2}\frac{1}{2-\rho} + \frac{1}{3}\frac{1}{3-\rho} + \dots + \frac{1}{K}\frac{1}{K-\rho}$  is the K<sup>th</sup> partial sum of the sequence  $\left\langle \frac{1}{1-\rho}, \frac{1}{2}\frac{1}{2-\rho}, \dots, \frac{1}{K}\frac{1}{K-\rho}, \dots \right\rangle$ , and  $Sum_{K} = \frac{1}{1-\rho} + \frac{1}{2}\frac{1}{2-\rho} + \frac{1}{3}\frac{1}{3-\rho} + \dots + \frac{1}{K}\frac{1}{K-\rho}$  is the K<sup>th</sup> partial sum of the sequence  $\left\langle \frac{1}{1-\rho}, \frac{1}{2}\frac{1}{2-\rho}, \dots, \frac{1}{K}\frac{1}{K-\rho}, \dots \right\rangle$ ,

 $\mathsf{Sum}_{\mathsf{K}-\rho} = \frac{1}{1-\rho} + \frac{1}{2-\rho} + \frac{1}{3-\rho} + \dots + \frac{1}{\mathsf{K}-\rho}. \text{ is the } \mathsf{K}^{th} \text{ partial sum of the sequence } \Big\langle \frac{1}{1-\rho}, \frac{1}{2-\rho}, \dots, \frac{1}{\mathsf{K}-\rho}, \dots \Big\rangle.$ 

The mean response time bounds and approximation of  $P_N^K$  is equal to the mean response time bounds and approximation of  $P_K$ . The Markov state diagrams of  $P_N^K$  and  $P_K$  are, however, different. The difference between the Markov diagrams of  $P_N^K$  and  $P_K$  account for the differences in the throughput and the queue length of  $P_N^K$  and  $P_K$ . The throughput of  $P_N^K$  equals  $\lambda$  while the throughput of  $P_K$  equals  $\lambda * K/N$ . Subsequently, the queue length (*i.e.*, number of jobs) at  $P_N^K$  and  $P_K$  are different. The queue length can be computed using Little's Law.

We validate our model response time against simulated response times. The simulated response times are accurate within 1% at 95% confidence. Figures 6, 7, and 8 plots the model and simulated response times for  $(\rho * N)/K = 0.1, 0.5, 0.9$ . Each figure has three graphs that correspond to N = 10, 50, 100. In each case, the number of sub-tasks vary from K = 1, 2, ..., N. Since  $\lambda/\mu$  is held constant, the server utilization  $\rho$  varies as K varies.



Figure 6: Model and simulated mean response time values when K  $\leq$  N and  $(\rho*N)/K=0.1$ 



Figure 7: Model and simulated mean response time values when K  $\leq$  N and  $(\rho*{\rm N})/{\rm K}=0.5$ 



Figure 8: Model and simulated mean response time values when  $K \leq N$  and  $(\rho * N)/K = 0.9$ 

## 6 Conclusions

This paper derives mean response time bounds and approximations for M/M/1 N-server fork-join queues where arriving jobs divide into  $1 \le K \le N$  sub-tasks. This work is notable for two key reasons, namely: a) the response time bounds and approximation are computationally simple and are presented as close-form equations, and b) the paper shows that the mean response time of  $P_N^K$  (a N-server fork-join queue where jobs divide into K sub-tasks) with arrival rate  $\lambda$  and service rate  $\mu$  is equal to the mean response time of  $P_K$  (a K-server fork-join queue where jobs divide into K sub-tasks) with arrival rate  $\lambda \ast K/N$  and service rate  $\mu$ . The relative error in the approximation, as compared to simulated values, is less than 5% for N  $\le 100$  and  $\rho$  varying from 0.1 to 0.9. Moreover, the relative error in the approximation remains approximately constant for fixed  $\rho$  as N varies from 2 to 100.

There are several extensions of this work. One extension is to consider multiple-class workloads where the value of K is not constant across all jobs and where the service requirement  $\mu$  is different for each job class. Another interesting extension to this work is to consider fork-join queues with phase-type service time distributions and variable sub-tasks.

## A Appendix

The mean service rate at Serial<sub>k</sub> of the Serial\_P<sub>N</sub>\_model is dependent on the number of jobs at Serial<sub>k-1</sub>, ..., Serial<sub>2</sub>, and Serial<sub>1</sub>. By analysing the Markov diagram of P<sub>N</sub>, we find that the rate varies according to the following rule: if there is at least one job in service center Serial<sub>k-1</sub>, the service rate at Serial<sub>k</sub> equals  $\mu$ , else if there is at least one job in service center Serial<sub>k-2</sub>, the service rate at Serial<sub>k</sub> equals  $2\mu$ , else if there is at least one job in service center Serial<sub>k-3</sub>, the service rate at Serial<sub>k</sub> equals  $3\mu$ , ..., else if there is at least one job in service center Serial<sub>1</sub>, the service rate at Serial<sub>k-2</sub>, ..., Serial<sub>1</sub>, the service rate at Serial<sub>k</sub> equals  $(k - 1)\mu$ , else (if there are no jobs in Serial<sub>k-1</sub>, Serial<sub>k-2</sub>, ..., Serial<sub>1</sub>), the service rate at Serial<sub>k</sub>

		rho = 0.1			rho = 0.5			rho = 0.9	
k	apprRT	simRT	err%	apprRT	simRT	err%	apprRT	simRT	err%
2	0.16520	0.16504	0.09695	1.41667	1.43928	1.57092	11.65909	12.52336	6.90126
3	0.20096	0.20090	0.02987	1.68333	1.71854	2.04883	13.37338	14.50522	7.80298
4	0.22767	0.22770	0.01318	1.87976	1.92015	2.10348	14.64354	16.10673	9.08434
5	0.24899	0.24971	0.28833	2.03532	2.08383	2.32793	15.65329	17.58570	10.98853
6	0.26672	0.26787	0.42931	2.16411	2.22879	2.90202	16.49153	18.63964	11.52442
7	0.28191	0.28275	0.29708	2.27400	2.34337	2.96027	17.20816	19.69006	12.60484
8	0.29518	0.29666	0.49889	2.36983	2.44826	3.20350	17.83404	20.46240	12.84483
9	0.30697	0.30781	0.27290	2.45480	2.53071	2.99955	18.38959	21.02895	12.55108
10	0.31758	0.31766	0.02518	2.53111	2.61548	3.22579	18.88904	22.08428	14.46839
11	0.32721	0.32762	0.12514	2.60038	2.68423	3.12380	19.34269	22.77161	15.05787
12	0.33605	0.33566	0.11619	2.66378	2.73650	2.65741	19.75823	22.94005	13.87015
13	0.34419	0.34465	0.13347	2.72224	2.79350	2.55092	20.14157	23.47786	14.21037
14	0.35176	0.35145	0.08821	2.77648	2.85122	2.62133	20.49735	23.71785	13.57838
15	0.35882	0.35921	0.10857	2.82705	2.90611	2.72048	20.82927	23.90498	12.86640
16	0.36544	0.36526	0.04928	2.87443	2.95269	2.65046	21.14032	24.36873	13.24817
17	0.37166	0.36982	0.49754	2.91899	2.99210	2.44343	21.43298	24.76445	13.45263
18	0.37754	0.37621	0.35353	2.96106	3.03529	2.44557	21.70929	25.07899	13.43635
19	0.38311	0.38318	0.01827	3.00089	3.07339	2.35896	21.97100	25.29888	13.15426
20	0.38840	0.38798	0.10825	3.03871	3.10882	2.25520	22.21956	25.58520	13.15464
21	0.39344	0.39286	0.14764	3.07471	3.13841	2.02969	22.45623	25.90421	13.31050
22	0.39825	0.39785	0.10054	3.10907	3.16828	1.86884	22.68210	26.31574	13.80786
23	0.40285	0.40155	0.32375	3.14192	3.20728	2.03786	22.89812	26.60038	13.91807
24	0.40725	0.40631	0.23135	3.17339	3.23581	1.92904	23.10510	26.64745	13.29339
25	0.41148	0.41004	0.35119	3.20359	3.27767	2.26014	23.30377	26.73399	12.83093
26	0.41555	0.41360	0.47147	3.23263	3.30973	2.32950	23.49477	27.12726	13.39055
27	0.41947	0.41689	0.61887	3.26058	3.35315	2.76069	23.67868	27.16729	12.84121
28	0.42324	0.42010	0.74744	3.28753	3.37493	2.58968	23.85600	27.25337	12.46587
29	0.42689	0.42556	0.31253	3.31354	3.39886	2.51025	24.02719	27.31246	12.02847
30	0.43041	0.43013	0.06510	3.33868	3.42525	2.52741	24.19265	27.88652	13.24608
31	0.43382	0.43265	0.27043	3.36301	3.45339	2.61714	24.35276	28.07468	13.25721
32	0.43713	0.43617	0.22010	3.38657	3.48081	2.70742	24.50786	28.31394	13.44242
33	0.44033	0.43985	0.10913	3.40941	3.50426	2.70671	24.65824	28.44451	13.31107
34	0.44344	0.44224	0.27135	3.43158	3.53273	2.86322	24.80419	28.51275	13.00667
35	0.44646	0.44526	0.26951	3.45311	3.55145	2.76901	24.94596	28.97331	13.90021
36	0.44940	0.44822	0.26326	3.4/404	3.5/518	2.82895	25.08378	29.06224	13.68945
3/	0.45225	0.45004	0.49107	3.49441	3.59181	2./11/2	25.21/86	29.09154	13.31549
38	0.45503	0.45229	0.60581	3.51423	3.61442	2.77195	25.34841	29.25261	13.34650
39	0.45774	0.45580	0.42563	3.33333	3.63170	2.70259	25.4/561	29.33367	13.15233
40	0.46039	0.45845	0.42755	3.35238	3.03081	2.09011	25.59962	29.59205	13.49130
41	0.46296	0.40157	0.30115	3.57074	3.00808	2.00903	25.72060	29.79437	13.07295
42	0.40348	0.40300	0.39233	3.3880/	3.09102	2.188/3	25.83809	29.90492	13.39/19
43	0.40/94	0.40033	0.34323	3.00018	3.70003	2.09420	25.93403	20.12776	13.04233
44	0.47054	0.40924	0.23442	3.02329	3.12220	2.03/30	20.00074	30.13770	13.50804
43	0.47409	0.4/189	0.10933	3.04002	3.14412	2.19394	20.1/093	20 25 29 2	12.0546
40	0.47702	0.47707	0.09212	3.03039	3.70340	2.09304	20.20473	30.33382	13.40340
4/	0.477042	0.477668	0.03334	3.68808	3.778821	2.11131	20.39020	30.47955	13.41030
40	0.47943	0.47000	0.57091	3 70344	3.70021	2.04320	20.49330	30.02197	13.40100
+7 50	0.48139	0.48083	0.15509	3 718/10	3 81371	2.03233	20.33473	30.50077	12 83608
50	0.40570	0.40150	0.75071	5.71077	5.015/1	2.77070	20.07572	30.02320	12.03070

Table 1: Model and simulated mean response time values for  $N=2,\cdots,50$ 

k         apprRT         simRT         crr%         apprRT         simRT         crr%           51         0.48577         0.4852         0.46534         3.7324         3.83598         2.679113         30.72810         12.81282           52         0.48781         0.48530         0.49650         3.74717         3.85288         2.72965         2.6.8648         30.92968         12.70183         31.07346         12.88669           54         0.49176         0.48874         0.61792         3.77884         3.8811         2.71503         27.01783         31.07346         12.88669           55         0.49368         0.48990         0.77159         3.78952         3.89668         2.75003         27.16197         31.12176         12.72354           56         0.49924         0.49952         0.66946         3.8212         3.89688         1.70876         27.33746         3.10360         11.97366           50         0.50103         0.49929         0.70100         3.85440         3.91788         1.80681         27.67063         31.34717         1.98708           61         0.50279         0.49929         0.70100         3.85440         3.91787         1.84412         27.60681         31.51555         12			rho = 0.1			rho = 0.5			rho = 0.9	
51         0.48372         0.48332         0.43342         3.83598         2.67832         2.679113         3.072810         12.81228           52         0.48780         0.48540         0.49650         3.74771         3.85288         2.72965         26.88648         30.82968         1.27027           53         0.48800         0.48720         0.53366         3.76191         3.86712         2.72063         26.98002         30.9477         12.86866           54         0.49170         0.48874         0.61792         3.77584         3.80688         2.7303         71.112176         12.72354           56         0.49556         0.49170         0.78503         3.80295         3.89999         2.48821         27.25049         31.03390         12.19122           57         0.49742         0.49302         0.66664         3.8912         3.89688         1.73384         7.4223         31.12450         11.89279           58         0.49929         0.7100         3.85440         3.91788         1.80681         2.76084         31.34717         11.87786           61         0.50452         0.50512         0.81702         3.89080         3.95517         1.82714         2.78986         31.47171         1.827251	k	apprRT	simRT	err%	apprRT	simRT	err%	apprRT	simRT	err%
52         0.48810         0.48950         3.74771         3.85288         2.72965         26.88648         30.82968         12.79027           53         0.48980         0.48720         0.53366         3.76191         3.86712         2.72063         26.98002         30.96477         12.88649           54         0.49176         0.48890         0.71759         3.77843         3.88111         2.71237         2.7.0183         31.07436         12.88049           55         0.49365         0.49707         0.78503         3.80295         3.89999         2.4881         27.25049         31.03436         1.19736           56         0.49707         0.75503         3.80295         3.89999         2.4881         27.25094         31.24451         11.95768           59         0.50103         0.49793         0.62258         3.84187         390898         1.70657         27.50694         31.24471         11.98768           61         0.50452         0.50050         0.72831         3.84187         3997788         1.86412         27.95076         31.60768         12.20248           62         0.50055         0.5012         0.87702         3.97734         0.327781         3.8412         27.95076         31.60768	51	0.48577	0.48352	0.46534	3.73324	3.83598	2.67832	26.79113	30.72810	12.81228
53         0.48920         0.48720         0.53366         3.76191         3.86712         2.72063         26,98002         30.94747         12.88646           54         0.49176         0.48874         0.61792         3.77584         3.88111         2.71237         27.07183         31.07436         12.8849           55         0.49356         0.49970         0.78503         3.80295         3.80999         2.14812         27.25049         31.03607         11.97386           56         0.49924         0.49592         0.66646         3.8118         3.90688         1.73883         27.42293         31.12450         11.89279           59         0.50103         0.49793         0.62228         3.84148         3.90688         1.73681         27.50563         31.34717         11.98708           61         0.50452         0.50080         0.74281         3.86003         3.95517         1.62749         27.85964         31.34580         12.20248           62         0.50125         0.870702         3.90255         3.95517         1.62749         27.89764         31.82206         12.06495           63         0.51148         0.50833         0.87543         3.99051         4.06719         2.32891         2.32011 <td>52</td> <td>0.48781</td> <td>0.48540</td> <td>0.49650</td> <td>3.74771</td> <td>3.85288</td> <td>2.72965</td> <td>26.88648</td> <td>30.82968</td> <td>12.79027</td>	52	0.48781	0.48540	0.49650	3.74771	3.85288	2.72965	26.88648	30.82968	12.79027
54         0.49176         0.48874         0.61792         3.77584         3.88111         2.71237         27.07183         3.112176         12.78349           55         0.49368         0.48990         0.77159         3.78952         3.89099         2.48821         27.25049         31.12176         12.72349           56         0.49556         0.49170         0.78503         3.80295         3.89099         2.48821         27.25049         31.01450         11.97386           57         0.49742         0.49366         0.76166         3.81418         3.90088         1.70676         27.50694         31.24451         11.96768           59         0.50103         0.49793         0.62258         3.84187         3.90888         1.70676         27.50644         31.12450         11.89279           50         0.50125         0.81051         3.86673         3.97378         1.80412         27.0506         31.34711         11.98708           61         0.50455         0.50512         0.78131         3.89080         3.95257         1.84412         27.05063         31.47116         12.2044           63         0.50195         0.50351         0.87723         9.97537         1.84412         27.90883         31.79126 </td <td>53</td> <td>0.48980</td> <td>0.48720</td> <td>0.53366</td> <td>3.76191</td> <td>3.86712</td> <td>2.72063</td> <td>26.98002</td> <td>30.96477</td> <td>12.86866</td>	53	0.48980	0.48720	0.53366	3.76191	3.86712	2.72063	26.98002	30.96477	12.86866
55         0.49368         0.48990         0.77159         3.78952         3.89068         2.7503         27.16197         31.12176         1.27234           56         0.49974         0.49366         0.76166         3.81615         3.90009         2.15226         27.33744         31.03300         12.1716           57         0.49742         0.49366         0.76166         3.80009         2.15226         27.33744         31.05607         11.97386           58         0.49924         0.49592         0.66946         3.82123         3.90688         1.73883         27.42233         31.12461         11.89708           60         0.50279         0.49929         0.70100         3.85401         3.9169         1.66569         27.58956         31.34717         11.89708           61         0.50422         0.50080         0.74281         3.88643         3.93788         1.80681         27.67081         31.5185         1.20048           63         0.50790         0.50355         0.78712         3.92255         1.806112         27.9313         31.82286         1.20244         31.97584         1.221712         27.8313         31.82286         1.20248           64         0.51575         0.32768         3.97584	54	0.49176	0.48874	0.61792	3.77584	3.88111	2.71237	27.07183	31.07436	12.88049
56         0.49556         0.49170         0.78503         3.80295         3.89999         2.48821         27.25049         31.03500         12.19122           57         0.49742         0.49366         0.76166         3.81615         3.90009         2.15226         27.33746         31.05607         11.97386           58         0.49924         0.49592         0.66944         3.82912         3.89688         1.73883         27.42293         31.12450         11.89279           59         0.50103         0.49793         0.62228         3.84187         3.90888         1.70676         27.50694         31.24717         11.98708           61         0.50452         0.50015         0.81051         3.87886         3.95266         1.86710         27.5076         31.60768         1.220448           63         0.50955         0.50512         0.87702         3.99255         3.97587         1.84412         27.90688         31.79126         12.20449           64         0.5178         0.50833         0.87542         3.94779         4.02072         2.08893         28.13220         3.203017         12.16968           65         0.51781         0.51305         0.55745         3.94779         4.02022         2.08976 <td>55</td> <td>0.49368</td> <td>0.48990</td> <td>0.77159</td> <td>3.78952</td> <td>3.89668</td> <td>2.75003</td> <td>27.16197</td> <td>31.12176</td> <td>12.72354</td>	55	0.49368	0.48990	0.77159	3.78952	3.89668	2.75003	27.16197	31.12176	12.72354
57         0.49742         0.49952         0.66946         3.82912         3.89688         1.73883         27.42293         31.12450         11.89279           59         0.50103         0.49793         0.62284         3.84187         3.90688         1.70676         27.50954         31.2450         11.89279           50         0.50103         0.49793         0.62284         3.84187         3.90688         1.70676         27.50954         31.2451         11.95768           60         0.50279         0.49929         0.70100         3.85473         3.93788         1.80651         27.67081         31.51585         12.20034           61         0.50952         0.50512         0.81051         3.87863         3.95517         1.62749         27.82944         31.75880         12.37281           64         0.50955         0.50512         0.87702         3.90255         3.97864         1.62191         27.98131         31.82286         12.00494           65         0.5118         0.50566         0.91203         3.91411         3.97864         1.62191         2.93131         31.82286         12.06498           66         0.51574         0.95903         3.94779         4.03205         2.08972         28.05822	56	0.49556	0.49170	0.78503	3.80295	3.89999	2.48821	27.25049	31.03390	12.19122
58         0.49924         0.49592         0.69946         3.82912         3.89688         1.73883         27.42293         31.12450         11.89279           59         0.50103         0.49793         0.62258         3.84187         3.90858         1.70676         27.50694         31.24641         11.96768           60         0.50279         0.49929         0.70100         3.85440         3.91969         1.66569         27.58595         31.34717         11.98708           61         0.50452         0.50080         0.74281         3.86786         3.95266         1.86710         27.75076         31.60768         12.20048           63         0.50790         0.50351         0.87702         3.90255         3.97587         1.8412         27.90688         31.97126         12.21839           65         0.51178         0.50656         0.91203         3.94717         4.02025         2.08803         28.1320         32.00317         12.16968           66         0.51744         0.51957         0.32768         3.95684         4.02015         2.08897         28.1320         32.0017         12.16968           68         0.51744         0.51903         0.01541         3.95684         4.02015         2.05813	57	0.49742	0.49366	0.76166	3.81615	3.90009	2.15226	27.33746	31.05607	11.97386
59         0.50103         0.49793         0.62258         3.84187         3.90888         1.70676         27.50694         31.24641         11.96768           60         0.50279         0.49929         0.70100         3.85440         3.91969         1.66569         27.58956         31.34717         11.98708           61         0.50422         0.50080         0.74281         3.86673         3.93788         1.80611         27.75076         31.60768         12.20248           63         0.50790         0.50395         0.78381         3.89080         3.95517         1.62749         27.82944         31.75880         12.37251           64         0.50955         0.50512         0.87742         3.92551         4.00802         2.05862         28.0582         31.97126         12.21839           65         0.51128         0.50949         0.55393         3.94779         4.02072         2.08893         28.13220         32.0017         12.16968           6         0.5194         0.51935         0.51903         0.01541         3.99484         4.04042         2.15915         28.20201         32.0221         12.18932           70         0.51895         0.51903         0.01541         3.99444         4.054068	58	0.49924	0.49592	0.66946	3.82912	3.89688	1.73883	27.42293	31.12450	11.89279
60         0.50279         0.49929         0.70100         3.8440         3.91969         1.66569         27.58956         31.34717         11.98708           61         0.50452         0.50080         0.74281         3.86673         3.93788         1.80681         27.67081         31.51585         12.20034           62         0.50522         0.50512         0.81051         3.87886         3.95266         1.86710         27.75076         31.60768         12.20248           63         0.50700         0.50355         0.78381         3.89080         3.95517         1.627149         27.82944         31.75880         12.37251           64         0.50572         0.50512         0.87702         3.90255         3.97587         1.84412         27.90688         31.97126         12.20498           65         0.51118         0.50565         0.91230         3.92561         4.00802         2.05862         28.05822         31.95715         12.2049           67         0.5144         0.51575         0.32768         3.95868         4.04040         2.15915         28.27691         32.20213         12.18932           70         0.51840         0.51931         0.69264         3.99045         4.06068         1.94684 <td>59</td> <td>0.50103</td> <td>0.49793</td> <td>0.62258</td> <td>3.84187</td> <td>3.90858</td> <td>1.70676</td> <td>27.50694</td> <td>31.24641</td> <td>11.96768</td>	59	0.50103	0.49793	0.62258	3.84187	3.90858	1.70676	27.50694	31.24641	11.96768
61         0.50452         0.50080         0.74281         3.8673         3.93788         1.80681         27.67081         31.51585         12.20034           62         0.50790         0.50395         0.8381         3.89080         3.95216         1.86710         27.75076         31.60768         12.20248           63         0.50790         0.50395         0.78381         3.89080         3.95517         1.62749         27.82944         31.75880         12.37251           64         0.50955         0.50512         0.87702         3.99253         3.97587         1.84412         27.90688         31.79126         12.21839           66         0.51128         0.50833         0.87542         3.92551         4.00802         2.08893         28.13220         32.03017         12.16968           66         0.51130         0.51545         0.39477         4.03205         2.08976         28.20508         32.16927         12.32291           67         0.51895         0.51903         0.01541         3.9642         4.05400         2.08633         28.4770         32.3649         12.33526           71         0.52145         0.51903         0.01541         3.9642         4.06400         2.08633         28.34770	60	0.50279	0.49929	0.70100	3.85440	3.91969	1.66569	27.58956	31.34717	11.98708
62         0.50622         0.5015         0.81051         3.87886         3.95266         1.86710         27.75076         31.60768         12.20248           63         0.50790         0.50395         0.50512         0.87702         3.90285         3.97587         1.62749         27.82944         31.79126         12.21839           65         0.51118         0.50656         0.91203         3.91411         3.97864         1.62191         27.98313         31.82286         12.06595           66         0.51278         0.50833         0.87542         3.92551         4.00802         2.05862         28.05203         31.95715         12.2049           67         0.51435         0.50949         0.95390         3.93673         4.00720         2.08897         28.13220         32.0017         12.16968           68         0.51591         0.51305         0.55745         3.94779         4.03205         2.08976         28.20508         32.16927         12.33267           70         0.51903         0.01541         3.99945         4.06968         1.94684         28.48633         32.30628         11.82417           73         0.52478         0.52160         0.60966         4.00075         4.09951         2.19435	61	0.50452	0.50080	0.74281	3.86673	3.93788	1.80681	27.67081	31.51585	12.20034
63         0.50790         0.50395         0.78381         3.89080         3.95517         1.62749         27.82944         31.7580         12.37251           64         0.50955         0.50512         0.87702         3.90255         3.97587         1.84412         27.9688         31.79126         12.21839           65         0.51118         0.50656         0.91203         3.94111         3.97864         1.62191         7.98313         31.82286         12.06595           66         0.51278         0.50833         0.87542         3.92551         4.00802         2.08863         28.13220         32.0017         12.16968           67         0.51435         0.50745         0.32768         3.94674         4.03604         2.15915         82.7691         32.20213         12.18932           70         0.51895         0.51903         0.01541         3.96842         4.06404         2.15918         28.41750         32.26213         12.33261           71         0.52335         0.52006         0.63262         4.00075         4.04604         2.1528         28.41750         32.6564         11.95087           74         0.52478         0.51060         0.60966         4.01091         4.044455         0.8174	62	0.50622	0.50215	0.81051	3.87886	3.95266	1.86710	27.75076	31.60768	12.20248
64         0.50955         0.5012         0.87702         3.90255         3.97787         1.84112         2.790688         3.179126         1.21839           65         0.51118         0.50656         0.91203         3.91411         3.97864         1.62191         27.98313         31.82286         12.06595           66         0.51278         0.50833         0.87542         3.92551         4.00802         2.08893         28.13220         32.03017         12.16968           66         0.51591         0.51035         0.55745         3.94779         4.03205         2.08976         28.20508         32.16927         12.32291           70         0.51895         0.51903         0.01541         3.96442         4.05400         2.08633         28.34770         32.33649         12.33526           71         0.52044         0.51903         0.01541         3.96942         4.05400         2.08633         32.30628         11.893173           73         0.52355         0.52060         0.63262         4.00075         4.05484         28.48533         32.3664         11.905087           74         0.52478         0.52160         0.60966         4.01091         4.04455         0.83174         28.65422         32.49877 </td <td>63</td> <td>0.50790</td> <td>0.50395</td> <td>0.78381</td> <td>3.89080</td> <td>3.95517</td> <td>1.62749</td> <td>27.82944</td> <td>31.75880</td> <td>12.37251</td>	63	0.50790	0.50395	0.78381	3.89080	3.95517	1.62749	27.82944	31.75880	12.37251
65         0.51118         0.50656         0.91203         3.91411         3.97864         1.62191         27.98313         31.82286         12.06595           66         0.51278         0.50833         0.87542         3.92551         4.00802         2.05862         28.05822         31.95715         12.20049           67         0.51143         0.50949         0.95390         3.93673         4.00207         2.08893         28.13220         32.0017         12.16968           68         0.51591         0.51305         0.55745         3.94779         4.03205         2.08633         28.34770         3.20213         12.18932           70         0.51940         0.51921         0.323690         3.96042         4.04604         2.15288         28.41750         32.20213         12.189031           72         0.52140         0.51921         0.32360         3.98001         4.06758         2.15288         28.45433         32.30628         11.82417           73         0.52181         0.52101         0.68687         4.00051         2.19435         28.55422         32.42887         11.95087           74         0.52478         0.52100         0.60966         4.01091         4.04556         0.7076         28.75246 </td <td>64</td> <td>0.50955</td> <td>0.50512</td> <td>0.87702</td> <td>3.90255</td> <td>3.97587</td> <td>1.84412</td> <td>27.90688</td> <td>31.79126</td> <td>12.21839</td>	64	0.50955	0.50512	0.87702	3.90255	3.97587	1.84412	27.90688	31.79126	12.21839
66         0.51278         0.50833         0.87542         3.92551         4.00802         2.05862         28.05822         31.95715         12.20049           67         0.51435         0.50949         0.95390         3.93673         4.02072         2.08893         28.1322         32.0317         12.16968           68         0.51591         0.51305         0.55745         3.94779         4.03205         2.08976         28.20508         32.16927         12.32291           69         0.51895         0.51003         0.01541         3.96042         4.06404         2.15915         28.27691         32.20213         12.18932           70         0.52044         0.51921         0.23600         3.98001         4.06758         2.15288         28.41703         32.25241         11.89031           72         0.52190         0.51813         0.69264         3.99045         4.04955         0.83174         28.62118         32.50519         11.92487           73         0.52478         0.52160         0.60966         4.01091         4.044455         0.83174         28.62118         32.50519         11.9287           74         0.52478         0.52439         0.48633         4.00458         4.07151         0.75796 <td>65</td> <td>0.51118</td> <td>0.50656</td> <td>0.91203</td> <td>3.91411</td> <td>3.97864</td> <td>1.62191</td> <td>27.98313</td> <td>31.82286</td> <td>12.06595</td>	65	0.51118	0.50656	0.91203	3.91411	3.97864	1.62191	27.98313	31.82286	12.06595
67         0.51435         0.50949         0.95390         3.93673         4.02072         2.08893         28.13220         32.03017         12.16968           68         0.51591         0.51305         0.55745         3.94779         4.03205         2.08976         28.20508         32.16927         12.32291           69         0.51744         0.51575         0.32768         3.99888         4.04604         2.15915         28.27691         32.20213         12.18932           70         0.51895         0.51903         0.01541         3.96942         4.05400         2.08633         28.34770         32.203649         12.33526           71         0.52044         0.51921         0.23690         3.93001         4.06758         2.15288         28.41750         32.23649         12.33526           73         0.52478         0.52160         0.60966         4.01091         4.04455         0.83174         28.62118         32.50519         11.94889           75         0.52478         0.52130         0.36687         4.0203         4.0414         0.57392         28.68726         32.59632         11.95924           76         0.52439         0.48633         4.04052         4.08450         0.83927         28.8034 <td>66</td> <td>0.51278</td> <td>0.50833</td> <td>0.87542</td> <td>3.92551</td> <td>4.00802</td> <td>2.05862</td> <td>28.05822</td> <td>31.95715</td> <td>12.20049</td>	66	0.51278	0.50833	0.87542	3.92551	4.00802	2.05862	28.05822	31.95715	12.20049
68         0.51591         0.51745         3.94779         4.03205         2.08976         28.20508         32.16927         12.32291           69         0.51744         0.51575         0.32768         3.95868         4.04604         2.15915         28.27691         32.20213         12.38326           70         0.52044         0.51921         0.23690         3.96041         4.06758         2.15288         28.41750         32.33649         12.33526           71         0.52140         0.51831         0.69264         3.99045         4.06668         1.94684         28.48633         32.30628         11.82417           73         0.52335         0.52060         0.63262         4.00075         4.09051         2.19435         28.55422         32.42987         11.95087           74         0.52478         0.52110         0.58687         4.02093         4.04414         0.57392         28.68726         32.50519         11.94889           75         0.52618         0.52311         0.58687         4.02093         4.04584         0.70796         28.86713         32.65664         11.95524           76         0.52618         0.523140         0.46633         4.04573         4.09528         0.88037         28.84337	67	0.51435	0.50949	0.95390	3.93673	4.02072	2.08893	28.13220	32.03017	12.16968
69         0.51744         0.51575         0.32768         3.95868         4.04604         2.15915         28.27691         32.20213         12.18932           70         0.51895         0.51903         0.01541         3.96942         4.05400         2.08633         28.34770         32.33649         12.33526           71         0.52044         0.51921         0.23600         3.99045         4.06968         1.94684         28.43750         32.25241         11.89031           72         0.52190         0.51831         0.69264         3.99045         4.06968         1.94684         28.4633         32.30628         11.82417           73         0.52375         0.52160         0.63262         4.00075         4.09051         2.19435         28.85422         32.42987         11.95087           74         0.52478         0.52161         0.52637         0.52643         4.04058         4.07151         0.75967         28.681681         32.71402         11.91297           78         0.53030         0.52788         0.45844         4.05022         4.08450         0.83927         28.81681         32.71402         11.91297           78         0.53164         0.52987         0.34044         4.0573         4.09528 <td>68</td> <td>0.51591</td> <td>0.51305</td> <td>0.55745</td> <td>3.94779</td> <td>4.03205</td> <td>2.08976</td> <td>28.20508</td> <td>32.16927</td> <td>12.32291</td>	68	0.51591	0.51305	0.55745	3.94779	4.03205	2.08976	28.20508	32.16927	12.32291
70         0.51895         0.51903         0.01541         3.96942         4.05400         2.08633         28.34770         3.233649         12.33526           71         0.52044         0.51921         0.23690         3.98001         4.06758         2.15288         2.841750         32.25241         11.89031           72         0.52130         0.51831         0.69264         3.99045         4.06068         1.94684         28.48633         32.30628         11.82417           73         0.52478         0.52160         0.60966         4.01091         4.04455         0.83174         28.55422         32.42987         11.95087           74         0.52478         0.52160         0.60966         4.01091         4.04455         0.83174         28.62118         32.50519         11.94889           75         0.52639         0.48633         4.04058         4.07151         0.75967         28.81681         32.71402         11.91297           78         0.53030         0.52788         0.48633         4.04058         4.07151         0.75967         28.81681         32.71402         11.91297           78         0.53046         0.53142         0.39931         4.08573         4.10453         0.91074         29.00501<	69	0.51744	0.51575	0.32768	3.95868	4.04604	2.15915	28.27691	32.20213	12.18932
71         0.52044         0.51921         0.23690         3.98001         4.06758         2.15288         28.41750         32.25241         11.89031           72         0.52190         0.51831         0.69264         3.99045         4.06968         1.94684         28.48633         32.30628         11.82417           73         0.52335         0.52060         0.63262         4.00075         4.09051         2.19435         28.55422         32.42987         11.95087           74         0.52618         0.52110         0.66066         4.01091         4.04455         0.83174         28.65118         32.59632         11.99234           75         0.52618         0.52131         0.58687         4.02093         4.0414         0.57392         28.68726         32.59632         11.99234           76         0.52757         0.52434         0.61601         4.03082         4.05956         0.70796         28.75246         32.65664         11.9524           77         0.52895         0.52639         0.43844         4.05022         4.08450         0.83927         28.8034         32.84465         12.06964           80         0.53296         0.53142         0.28979         4.06913         4.10553         0.91074	70	0.51895	0.51903	0.01541	3.96942	4.05400	2.08633	28.34770	32.33649	12.33526
72         0.52190         0.51831         0.69264         3.99045         4.06968         1.94684         28.48633         32.30628         11.82417           73         0.52335         0.52006         0.63262         4.00075         4.09051         2.19435         28.55422         32.42987         11.95087           74         0.52478         0.52160         0.60966         4.01091         4.04455         0.83174         28.62118         32.5519         11.94889           75         0.52618         0.52311         0.58687         4.02093         4.04414         0.57392         28.68726         32.56664         11.9524           76         0.52757         0.52434         0.61601         4.03082         4.05956         0.70796         28.75246         32.65664         11.9524           77         0.52895         0.52639         0.48633         4.04058         4.07151         0.75967         28.8407         32.94905         12.160694           79         0.53164         0.52987         0.3404         4.05973         4.09528         0.86807         28.94307         32.94905         12.18303           80         0.53426         0.53448         0.40913         4.10653         0.91074         29.00501	71	0.52044	0.51921	0.23690	3.98001	4.06758	2.15288	28.41750	32.25241	11.89031
730.523350.520060.632624.000754.090512.1943528.5542232.4298711.95087740.524780.521600.609664.010914.044550.8317428.6211832.5051911.94889750.526180.523110.586874.020934.044140.5739228.6872632.566411.95224760.527570.524340.616014.030824.059560.7079628.8168132.7140211.91297780.530300.527880.458444.050224.084500.8392728.8803432.8445612.06964790.531640.529870.334044.059734.095280.8680728.9430732.9490512.15810800.532960.531420.289794.069134.106530.9107429.0050133.0377112.20635810.534260.532440.341824.078404.123661.0975729.0661833.0798712.1333820.535550.533420.399314.087574.140531.2700629.1266133.1818112.22115830.536820.535370.506194.105574.152001.1182629.2452933.654412.34526850.539320.536600.506904.114414.150700.8743129.3035833.4335412.35275860.540550.537880.623174.140324.165790.8114129.4181533.6028412.44218890.541760.	72	0.52190	0.51831	0.69264	3.99045	4.06968	1.94684	28.48633	32.30628	11.82417
74         0.52478         0.52160         0.60966         4.01091         4.04455         0.83174         28.62118         32.50519         11.94889           75         0.52618         0.52311         0.58687         4.02093         4.04414         0.57392         28.68726         32.59632         11.99234           76         0.522757         0.52434         0.61601         4.03082         4.05956         0.70796         28.75246         32.6664         11.95524           77         0.52895         0.52639         0.48633         4.04058         4.07151         0.75967         28.84031         32.71402         11.91297           78         0.53030         0.52788         0.45844         4.05022         4.08450         0.83927         28.84031         32.84456         12.06964           79         0.53164         0.52987         0.33404         4.05973         4.09528         0.86807         28.94307         32.94905         12.15810           80         0.53255         0.53342         0.39931         4.08757         4.10236         1.09757         29.0618         33.07987         12.13333           82         0.53468         0.40024         4.09662         4.15241         1.34356         29.18630 </td <td>73</td> <td>0.52335</td> <td>0.52006</td> <td>0.63262</td> <td>4.00075</td> <td>4.09051</td> <td>2.19435</td> <td>28.55422</td> <td>32.42987</td> <td>11.95087</td>	73	0.52335	0.52006	0.63262	4.00075	4.09051	2.19435	28.55422	32.42987	11.95087
750.526180.523110.586874.020934.044140.5739228.6872632.5963211.99234760.527570.524340.616014.030824.059560.7079628.7524632.6566411.95524770.528950.526390.486334.040584.071510.7596728.8168132.7140211.91297780.530300.527880.458444.050224.084500.8392728.8803432.8445612.06964790.531640.529870.334044.059734.095280.8680728.9430732.9490512.15810800.532260.531420.289794.069134.106530.9107429.0050133.0377112.20635810.534260.532440.341824.078404.123661.0975729.0661833.0798712.13333820.535550.533420.399314.087574.140531.2790629.1266133.1818112.22115830.536820.534680.400244.096624.152411.3435629.1863033.2609012.25042840.538080.535370.506904.114414.150700.8743129.3015833.4335412.35275860.540550.537880.496394.123154.167971.0753429.3612033.5060612.35620870.541760.539870.572364.140324.160980.4965229.4744533.6628412.44218890.54415 <td< td=""><td>74</td><td>0.52478</td><td>0.52160</td><td>0.60966</td><td>4.01091</td><td>4.04455</td><td>0.83174</td><td>28.62118</td><td>32.50519</td><td>11.94889</td></td<>	74	0.52478	0.52160	0.60966	4.01091	4.04455	0.83174	28.62118	32.50519	11.94889
760.527570.524340.616014.030824.059560.7079628.7524632.6566411.95524770.528950.526390.486334.040584.071510.7596728.8168132.7140211.91297780.530300.527880.458444.050224.084500.8392728.8803432.8445612.06964790.531640.529870.334044.059734.095280.8680728.9430732.9490512.15810800.532960.531420.289794.069134.106530.9107429.0050133.0377112.20635810.532420.339314.087574.140531.2790629.1266133.1818112.22115830.536820.534480.400244.096624.152411.3435629.1863033.2609012.25042840.538080.535370.506194.105574.152001.1182629.2452933.3654412.34856850.539320.536600.506904.114414.150700.8743129.303583.4335412.35275860.540550.537880.496394.123154.167971.0753429.3612033.5006012.35620870.541760.539020.508334.131794.165590.8114129.4181533.6016612.45031880.542960.539870.572364.148774.187530.9256129.5301233.5697612.03357900.544150.54078	75	0.52618	0.52311	0.58687	4.02093	4.04414	0.57392	28.68726	32.59632	11.99234
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	76	0.52757	0.52434	0.61601	4.03082	4.05956	0.70796	28.75246	32.65664	11.95524
78       0.53030       0.52788       0.45844       4.05022       4.08450       0.83927       28.88034       32.84456       12.06964         79       0.53164       0.52987       0.33404       4.05973       4.09528       0.86807       28.94307       32.94905       12.15810         80       0.53296       0.53142       0.28979       4.06913       4.10653       0.91074       29.00501       33.03771       12.20635         81       0.53246       0.53244       0.34182       4.07840       4.12366       1.09757       29.06618       33.07987       12.13333         82       0.53555       0.53342       0.39931       4.08757       4.14053       1.27906       29.12661       33.18181       12.22115         83       0.53682       0.53468       0.40024       4.09662       4.15241       1.34356       29.18630       33.26090       12.25042         84       0.53808       0.53537       0.50619       4.10557       4.15200       1.11826       29.1261       33.36544       12.34856         85       0.53932       0.53633       4.12315       4.16797       1.07534       29.36120       33.50060       12.35275         86       0.54055       0.53788 <td< td=""><td>77</td><td>0.52895</td><td>0.52639</td><td>0.48633</td><td>4.04058</td><td>4.07151</td><td>0.75967</td><td>28.81681</td><td>32.71402</td><td>11.91297</td></td<>	77	0.52895	0.52639	0.48633	4.04058	4.07151	0.75967	28.81681	32.71402	11.91297
79       0.53164       0.52987       0.33404       4.05973       4.09528       0.86807       28.94307       32.94905       12.15810         80       0.53296       0.53142       0.28979       4.06913       4.10653       0.91074       29.00501       33.03771       12.20355         81       0.53426       0.53244       0.34182       4.07840       4.12366       1.09757       29.06618       33.07987       12.13333         82       0.53555       0.53342       0.39931       4.08757       4.14053       1.27906       29.12661       33.18181       12.22115         83       0.53682       0.53468       0.40024       4.09662       4.15241       1.34356       29.18630       33.26090       12.25042         84       0.53808       0.53537       0.50619       4.10557       4.15200       1.11826       29.18630       33.26090       12.25042         84       0.53932       0.53660       0.50690       4.11441       4.15070       0.87431       29.30120       33.6544       12.34856         85       0.54055       0.53788       0.49639       4.12315       4.16797       1.07534       29.36120       33.60166       12.45031         86       0.54415 <td< td=""><td>78</td><td>0.53030</td><td>0.52788</td><td>0.45844</td><td>4.05022</td><td>4.08450</td><td>0.83927</td><td>28.88034</td><td>32.84456</td><td>12.06964</td></td<>	78	0.53030	0.52788	0.45844	4.05022	4.08450	0.83927	28.88034	32.84456	12.06964
800.532960.531420.289794.069134.106530.9107429.0050133.0377112.20635810.534260.532440.341824.078404.123661.0975729.0661833.0798712.13333820.535550.533420.399314.087574.140531.2790629.1266133.1818112.22115830.536820.534680.400244.096624.152411.3435629.1863033.2609012.25042840.538080.535370.506194.105574.152001.1182629.2452933.3654412.34856850.539320.536600.506904.114414.150700.8743129.305833.4335412.35275860.540550.537880.496394.123154.167971.0753429.3612033.5006012.35620870.541760.539020.508334.131794.165590.8114129.4181533.6016612.45031880.542960.539870.572364.140324.160980.4965229.4744533.6628412.44218890.544150.540780.623174.148774.187530.9256129.5301233.5697612.03357900.545320.541770.655264.157124.194120.8821929.5851733.659811.70694920.547630.544160.637684.173544.213080.9385129.6934733.6756211.82502930.548770	79	0.53164	0.52987	0.33404	4.05973	4.09528	0.86807	28.94307	32.94905	12.15810
81         0.53426         0.53244         0.34182         4.07840         4.12366         1.09757         29.06618         33.07987         12.13333           82         0.53555         0.53342         0.39931         4.08757         4.14053         1.27906         29.12661         33.18181         12.22115           83         0.53682         0.53468         0.40024         4.09662         4.15241         1.34356         29.18630         33.26090         12.25042           84         0.53808         0.53537         0.50619         4.10557         4.15200         1.11826         29.24529         33.36544         12.34856           85         0.53932         0.53660         0.50690         4.11441         4.15070         0.87431         29.30358         33.43354         12.35275           86         0.54055         0.53788         0.49639         4.12315         4.16797         1.07534         29.36120         33.50060         12.35620           87         0.54176         0.53902         0.50833         4.1179         4.16559         0.81141         29.41815         33.60241         12.44218           89         0.54415         0.54078         0.62317         4.14877         4.18753         0.92561 <td>80</td> <td>0.53296</td> <td>0.53142</td> <td>0.28979</td> <td>4.06913</td> <td>4.10653</td> <td>0.91074</td> <td>29.00501</td> <td>33.03771</td> <td>12.20635</td>	80	0.53296	0.53142	0.28979	4.06913	4.10653	0.91074	29.00501	33.03771	12.20635
82       0.53555       0.53342       0.39931       4.08757       4.14053       1.27906       29.12661       33.18181       12.22115         83       0.53682       0.53468       0.40024       4.09662       4.15241       1.34356       29.18630       33.26090       12.25042         84       0.53808       0.53537       0.50619       4.10557       4.15200       1.11826       29.24529       33.36544       12.34856         85       0.53932       0.53660       0.50690       4.11441       4.15070       0.87431       29.30358       33.43354       12.35275         86       0.54055       0.53788       0.49639       4.12315       4.16797       1.07534       29.36120       33.50060       12.35620         87       0.54176       0.53902       0.50833       4.13179       4.16559       0.81141       29.41815       33.66284       12.45031         88       0.54296       0.53987       0.57236       4.14032       4.16098       0.49652       29.47445       33.66284       12.44218         89       0.54415       0.54078       0.62317       4.14877       4.18753       0.92561       29.53012       33.56976       12.03357         90       0.54532 <t< td=""><td>81</td><td>0.53426</td><td>0.53244</td><td>0.34182</td><td>4.07840</td><td>4.12366</td><td>1.09757</td><td>29.06618</td><td>33.07987</td><td>12.13333</td></t<>	81	0.53426	0.53244	0.34182	4.07840	4.12366	1.09757	29.06618	33.07987	12.13333
83       0.53682       0.53468       0.40024       4.09662       4.15241       1.34356       29.18630       33.26090       12.25042         84       0.53808       0.53537       0.50619       4.10557       4.15200       1.11826       29.24529       33.36544       12.34856         85       0.53932       0.53660       0.50690       4.11441       4.15070       0.87431       29.30358       33.43354       12.35275         86       0.54055       0.53788       0.49639       4.12315       4.16797       1.07534       29.36120       33.50060       12.35620         87       0.54176       0.53902       0.50833       4.13179       4.16559       0.81141       29.41815       33.60166       12.45031         88       0.54296       0.53987       0.57236       4.14032       4.16098       0.49652       29.47445       33.66284       12.44218         89       0.54415       0.54078       0.62317       4.14877       4.18753       0.92561       29.53012       33.56958       11.70694         92       0.5448       0.54315       0.61309       4.16537       4.20692       0.98766       29.63961       33.56958       11.70694         92       0.54763 <td< td=""><td>82</td><td>0.53555</td><td>0.53342</td><td>0.39931</td><td>4.08757</td><td>4.14053</td><td>1.27906</td><td>29.12661</td><td>33.18181</td><td>12.22115</td></td<>	82	0.53555	0.53342	0.39931	4.08757	4.14053	1.27906	29.12661	33.18181	12.22115
84         0.53808         0.53537         0.50619         4.10557         4.15200         1.11826         29.24529         33.36544         12.34856           85         0.53932         0.53660         0.50690         4.11441         4.15070         0.87431         29.30358         33.43354         12.35275           86         0.54055         0.53788         0.49639         4.12315         4.16797         1.07534         29.36120         33.50060         12.35620           87         0.54176         0.53902         0.50833         4.13179         4.16559         0.81141         29.41815         33.60166         12.45031           88         0.54296         0.53987         0.57236         4.14032         4.16098         0.49652         29.47445         33.66284         12.44218           89         0.54415         0.54078         0.62317         4.14877         4.18753         0.92561         29.53012         33.56976         12.03357           90         0.54532         0.54177         0.65526         4.15712         4.19412         0.88219         29.58517         33.62919         12.02533           91         0.54648         0.54315         0.61309         4.16537         4.20692         0.98766 </td <td>83</td> <td>0.53682</td> <td>0.53468</td> <td>0.40024</td> <td>4.09662</td> <td>4.15241</td> <td>1.34356</td> <td>29.18630</td> <td>33.26090</td> <td>12.25042</td>	83	0.53682	0.53468	0.40024	4.09662	4.15241	1.34356	29.18630	33.26090	12.25042
85         0.53932         0.53660         0.50690         4.11441         4.15070         0.87431         29.30358         33.43354         12.35275           86         0.54055         0.53788         0.49639         4.12315         4.16797         1.07534         29.36120         33.50060         12.35620           87         0.54176         0.53902         0.50833         4.13179         4.16559         0.81141         29.41815         33.60166         12.45031           88         0.54296         0.53987         0.57236         4.14032         4.16098         0.49652         29.47445         33.66284         12.44218           89         0.54415         0.54078         0.62317         4.14877         4.18753         0.92561         29.53012         33.56976         12.03357           90         0.54532         0.54177         0.65526         4.15712         4.19412         0.88219         29.58517         33.62919         12.02533           91         0.54648         0.54315         0.61309         4.16537         4.20692         0.98766         29.69347         33.67562         11.82502           93         0.54763         0.54440         0.61052         4.18162         4.21775         0.85662 </td <td>84</td> <td>0.53808</td> <td>0.53537</td> <td>0.50619</td> <td>4.10557</td> <td>4.15200</td> <td>1.11826</td> <td>29.24529</td> <td>33.36544</td> <td>12.34856</td>	84	0.53808	0.53537	0.50619	4.10557	4.15200	1.11826	29.24529	33.36544	12.34856
86       0.54055       0.53788       0.49639       4.12315       4.16797       1.07534       29.36120       33.50060       12.35620         87       0.54176       0.53902       0.50833       4.13179       4.16559       0.81141       29.41815       33.60166       12.45031         88       0.54296       0.53987       0.57236       4.14032       4.16098       0.49652       29.47445       33.66284       12.44218         89       0.54415       0.54078       0.62317       4.14877       4.18753       0.92561       29.53012       33.56976       12.03357         90       0.54532       0.54177       0.65526       4.15712       4.19412       0.88219       29.58517       33.62919       12.02533         91       0.54648       0.54315       0.61309       4.16537       4.20692       0.98766       29.63961       33.56958       11.70694         92       0.54763       0.54416       0.63768       4.17354       4.21308       0.93851       29.69347       33.67562       11.82502         93       0.54877       0.54544       0.61052       4.18162       4.21775       0.85662       29.74674       33.71413       11.76774         94       0.54989 <t< td=""><td>85</td><td>0.53932</td><td>0.53660</td><td>0.50690</td><td>4.11441</td><td>4.15070</td><td>0.87431</td><td>29.30358</td><td>33.43354</td><td>12.35275</td></t<>	85	0.53932	0.53660	0.50690	4.11441	4.15070	0.87431	29.30358	33.43354	12.35275
87       0.54176       0.53902       0.50833       4.13179       4.16559       0.81141       29.41815       33.60166       12.45031         88       0.54296       0.53987       0.57236       4.14032       4.16098       0.49652       29.47445       33.66284       12.44218         89       0.54415       0.54078       0.62317       4.14877       4.18753       0.92561       29.53012       33.56976       12.03357         90       0.54532       0.54177       0.65526       4.15712       4.19412       0.88219       29.58517       33.62919       12.02533         91       0.54648       0.54315       0.61309       4.16537       4.20692       0.98766       29.63961       33.56958       11.70694         92       0.54763       0.54416       0.63768       4.17354       4.21308       0.93851       29.69347       33.67562       11.82502         93       0.54877       0.54544       0.61052       4.18162       4.21775       0.85662       29.74674       33.71413       11.76774         94       0.54989       0.54632       0.65346       4.18961       4.22130       0.75072       29.79945       33.80737       11.85517         95       0.55100 <t< td=""><td>86</td><td>0.54055</td><td>0.53788</td><td>0.49639</td><td>4.12315</td><td>4.16/9/</td><td>1.07534</td><td>29.36120</td><td>33.50060</td><td>12.35620</td></t<>	86	0.54055	0.53788	0.49639	4.12315	4.16/9/	1.07534	29.36120	33.50060	12.35620
88         0.54296         0.53987         0.57236         4.14032         4.16098         0.49652         29.47445         33.66284         12.44218           89         0.54415         0.54078         0.62317         4.14877         4.18753         0.92561         29.53012         33.56976         12.03357           90         0.54532         0.54177         0.65526         4.15712         4.19412         0.88219         29.58517         33.62919         12.02533           91         0.54648         0.54315         0.61309         4.16537         4.20692         0.98766         29.63961         33.56958         11.70694           92         0.54763         0.54416         0.63768         4.17354         4.21308         0.93851         29.69347         33.67562         11.82502           93         0.54877         0.54544         0.61052         4.18162         4.21775         0.85662         29.74674         33.71413         11.76774           94         0.54989         0.54632         0.65346         4.18961         4.22130         0.75072         29.79945         33.80737         11.85517           95         0.55100         0.54755         0.63008         4.19752         4.26127         1.26433 </td <td>87</td> <td>0.54176</td> <td>0.53902</td> <td>0.50833</td> <td>4.13179</td> <td>4.16559</td> <td>0.81141</td> <td>29.41815</td> <td>33.60166</td> <td>12.45031</td>	87	0.54176	0.53902	0.50833	4.13179	4.16559	0.81141	29.41815	33.60166	12.45031
890.344150.340780.623174.148774.187530.9256129.5301233.5697612.03557900.545320.541770.655264.157124.194120.8821929.5851733.6291912.02533910.546480.543150.613094.165374.206920.9876629.6396133.5695811.70694920.547630.544160.637684.173544.213080.9385129.6934733.6756211.82502930.548770.545440.610524.181624.217750.8566229.7467433.7141311.76774940.549890.546320.653464.189614.221300.7507229.7994533.8073711.85517950.551000.547550.630084.197524.251271.2643329.8516034.1006212.46024960.552100.548850.592154.205354.260281.2893529.9032034.1458512.42508970.553190.550320.521514.213094.271671.3713629.9542834.2518612.54700980.554270.551980.414874.220764.264391.0231230.0048334.3164112.56419990.555340.552730.472204.228354.270910.9965130.0548734.3305412.454421000.556390.553530.516684.238664.273060.8705730.1044134.4373112.58199	88	0.54296	0.53987	0.57236	4.14032	4.16098	0.49652	29.47445	33.66284	12.44218
900.545320.541770.655264.157124.194120.8821929.5851733.6291912.02533910.546480.543150.613094.165374.206920.9876629.6396133.5695811.70694920.547630.544160.637684.173544.213080.9385129.6934733.6756211.82502930.548770.545440.610524.181624.217750.8566229.7467433.7141311.76774940.549890.546320.653464.189614.221300.7507229.7994533.8073711.85517950.551000.547550.630084.197524.251271.2643329.8516034.1006212.46024960.552100.548850.592154.205354.260281.2893529.9032034.1458512.42508970.553190.550320.521514.213094.271671.3713629.9542834.2518612.54700980.554270.551980.414874.220764.264391.0231230.0048334.3164112.56419990.555340.552730.472204.228354.270910.9965130.0548734.3305412.454421000.556390.553530.516684.235864.273060.8705730.1044134.4373112.58199	89	0.54415	0.54078	0.62317	4.148//	4.18/53	0.92561	29.53012	33.56976	12.03357
910.546480.543150.613094.165374.206920.9876629.6396133.5695811.70694920.547630.544160.637684.173544.213080.9385129.6934733.6756211.82502930.548770.545440.610524.181624.217750.8566229.7467433.7141311.76774940.549890.546320.653464.189614.221300.7507229.7994533.8073711.85517950.551000.547550.630084.197524.251271.2643329.8516034.1006212.46024960.552100.548850.592154.205354.260281.2893529.9032034.1458512.42508970.553190.550320.521514.213094.271671.3713629.9542834.2518612.54700980.554270.551980.414874.220764.264391.0231230.0048334.3164112.56419990.555340.552730.472204.228354.270910.9965130.0548734.3305412.454421000.556390.553530.516684.235864.273060.8705730.1044134.4373112.58199	90	0.54532	0.541//	0.65526	4.15/12	4.19412	0.88219	29.58517	33.62919	12.02533
92       0.34763       0.34416       0.63768       4.17354       4.21308       0.93851       29.69347       33.67562       11.82502         93       0.54877       0.54544       0.61052       4.18162       4.21775       0.85662       29.74674       33.71413       11.76774         94       0.54989       0.54632       0.65346       4.18961       4.22130       0.75072       29.79945       33.80737       11.85517         95       0.55100       0.54755       0.63008       4.19752       4.25127       1.26433       29.85160       34.10062       12.46024         96       0.55210       0.54885       0.59215       4.20535       4.26028       1.28935       29.90320       34.14585       12.42508         97       0.55319       0.55032       0.52151       4.21309       4.27167       1.37136       29.95428       34.25186       12.54700         98       0.55427       0.55198       0.41487       4.22076       4.26439       1.02312       30.00483       34.31641       12.56419         99       0.55534       0.55273       0.47220       4.22835       4.27091       0.99651       30.05487       34.33054       12.45442         100       0.55639       <	91	0.54648	0.54315	0.61309	4.16537	4.20692	0.98/66	29.63961	33.56958	11.70694
95         0.34877         0.34344         0.01052         4.18162         4.21775         0.85662         29.74674         33.71413         11.76774           94         0.54989         0.54632         0.65346         4.18961         4.22130         0.75072         29.79945         33.80737         11.85517           95         0.55100         0.54755         0.63008         4.19752         4.25127         1.26433         29.85160         34.10062         12.46024           96         0.55210         0.54885         0.59215         4.20535         4.26028         1.28935         29.90320         34.14585         12.42508           97         0.55319         0.55032         0.52151         4.21309         4.27167         1.37136         29.95428         34.25186         12.54700           98         0.55427         0.55198         0.41487         4.22076         4.26439         1.02312         30.00483         34.31641         12.56419           99         0.55534         0.55273         0.47220         4.22835         4.27091         0.99651         30.05487         34.33054         12.45442           100         0.55639         0.55353         0.51668         4.23586         4.27306         0.87057<	92	0.54/63	0.54416	0.63/68	4.1/554	4.21308	0.93851	29.6934/	33.0/362	11.82502
94         0.34989 <th0.34141< th=""> <th0.341434< th=""> <th0.341< td=""><td>93</td><td>0.548//</td><td>0.54544</td><td>0.01052</td><td>4.18162</td><td>4.21//5</td><td>0.85062</td><td>29.74674</td><td>33./1413</td><td>11./0//4</td></th0.341<></th0.341434<></th0.34141<>	93	0.548//	0.54544	0.01052	4.18162	4.21//5	0.85062	29.74674	33./1413	11./0//4
95         0.33100         0.34755         0.05008         4.19752         4.25127         1.26433         29.85160         34.10062         12.46024           96         0.55210         0.54885         0.59215         4.20535         4.26028         1.28935         29.0320         34.14585         12.42508           97         0.55319         0.55032         0.52151         4.21309         4.27167         1.37136         29.95428         34.25186         12.54700           98         0.55427         0.55198         0.41487         4.22076         4.26439         1.02312         30.00483         34.31641         12.56419           99         0.55534         0.55273         0.47220         4.22835         4.27091         0.99651         30.05487         34.33054         12.45442           100         0.55639         0.55353         0.51668         4.23866         4.27306         0.87057         30.10441         34.43731         12.58199	94	0.54989	0.54032	0.03340	4.18961	4.22130	0.75072	29.79945	33.80/3/	11.8551/
90         0.33210         0.34885         0.39213         4.20335         4.20028         1.28935         29.90320         34.14585         12.42508           97         0.55319         0.55032         0.52151         4.21309         4.27167         1.37136         29.95428         34.25186         12.54700           98         0.55427         0.55198         0.41487         4.22076         4.26439         1.02312         30.00483         34.31641         12.56419           99         0.55534         0.55273         0.47220         4.22835         4.27091         0.99651         30.05487         34.33054         12.45442           100         0.55639         0.55353         0.51668         4.23586         4.27306         0.87057         30.10441         34.43731         12.58199	93	0.55100	0.54/55	0.03008	4.19/52	4.2312/	1.20433	29.85160	34.10062	12.46024
97         0.35319         0.35032         0.32131         4.21309         4.21107         1.37130         29.95428         34.25186         12.34700           98         0.55427         0.55198         0.41487         4.22076         4.26439         1.02312         30.00483         34.31641         12.56419           99         0.55534         0.55273         0.47220         4.22835         4.27091         0.99651         30.05487         34.33054         12.45442           100         0.55639         0.55353         0.51668         4.23586         4.27306         0.87057         30.10441         34.43731         12.58199	90	0.55210	0.54885	0.39213	4.20333	4.20028	1.20933	29.90320	34.14383	12.42308
99         0.55534         0.55273         0.47220         4.22835         4.27091         0.99651         30.05487         34.33054         12.45442           100         0.55639         0.55253         0.51668         4.23586         4.27306         0.87057         30.10441         34.43731         12.58199	97	0.553519	0.55052	0.32131	4.21309	4.2/10/	1.3/130	29.93428	34.23160	12.34700
100 0 55639 0 55353 0 51668 4 23586 4 27306 0 87057 30 10441 34 43731 12 58199	70 00	0.55427	0.55190	0.4140/	4.22070	4.20439	0.00651	30.00403	34.31041	12.30419
	100	0.555334	0.55275	0.47220	4.22033	4.27091	0.99031	30 104/1	34.33034	12.43442

Table 2: Model and simulated mean response time values for  $\mathsf{N}=51,\cdots,100$ 

equals  $k\mu$ . The service rate at server Serial<sub>1</sub> is equal to  $\mu$ . Thus, service rates at all queues Serial<sub>k</sub>,  $k = N, \dots, 2$  are dependent on the states of the queues Serial<sub>k-1</sub>,  $\dots$ , Serial<sub>2</sub>, and Serial<sub>1</sub>. Only queue Serial<sub>1</sub> is state independent.

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