http://www.cs.unh.edu/~ruml/cs758
Rod Cutting

- The Problem
- Optimal Value
- An Algorithm
- Solution Recovery
- Properties
- Substructure
- Break

2D DP
Given table of profits $p_i$ for each possible integer length $i$, find the best way to cut a rod of length $n$. Cuts are free, but must be of integer length.

<table>
<thead>
<tr>
<th>length $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>profit $p_i$</td>
<td>1</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>17</td>
<td>17</td>
<td>20</td>
<td>24</td>
<td>30</td>
</tr>
</tbody>
</table>

$\approx 2^{n-1}$ possible solutions! How to solve in $O(n^2)$ time?
Step 1: write down value of optimal solution

\[
\begin{align*}
\text{best}(n) &= \text{best profit achievable for length } n \\
\text{best}(n) &= \max_{first=1}^{n} (p_{first} + \text{best}(n - first)) \\
\text{best}(0) &= 0
\end{align*}
\]

What is the complexity of the naive recursive algorithm? How to make this efficient?
Step 2: compute optimal value (top-down or bottom-up)

1. best[0] ← 0
2. for len from 1 to $n$
3. best[len] ← $\max_{\text{first}=1}^{\text{len}} (p_{\text{first}} + \text{best}[\text{len} - \text{first}])$
4. best[$n$]

Will this access uninitialized data?
What is the complexity?
1. best[0] ← 0
2. cut[0] ← 0
3. for len from 1 to n
4.   best[len] ← −∞
5.   for first from 1 to len
6.     this ← p_{first} + best[len − first]
7.     if this > best[len]
8.       best[len] ← this
9.       cut[len] ← first
10. print best[n]
11. while n > 0
12. print cut[n]
13. n ← n − cut[n]
optimal substructure: global optimum uses optimal solutions of subproblems

ordering over subproblems: solve ‘smallest’ first, build ‘larger’ from them

‘overlapping’ subproblems: polynomial number of subproblems, each possibly used multiple times

independent subproblems: optimal solution of one subproblem doesn’t affect optimality of another
shortest path
- path to any intermediate vertex along optimal path must be optimal path to that vertex. otherwise, could be shorter.

longest simple path
- path to an intermediate vertex along optimal path may not use vertices used elsewhere: subproblems are not independent.
Break

Rod Cutting
- The Problem
- Optimal Value
- An Algorithm
- Solution Recovery
- Properties
- Substructure

Break

2D DP

- asst 4
- asst 5
- office hours
Two-Dimensional Dynamic Programming
Given two strings, \( x \) of length \( m \) and \( y \) of length \( n \), find a common (non-contiguous) subsequence that is as long as possible.

\[
x = \text{ABCBDAB}
\]
\[
y = \text{BDCABA}
\]
Given two strings, \( x \) of length \( m \) and \( y \) of length \( n \), find a common (non-contiguous) subsequence that is as long as possible.

\[
\begin{align*}
x &= \text{ABCBDAB} \\
y &= \text{BDCABA}
\end{align*}
\]

LCS = BCBA or BCAB

\[
\begin{align*}
x' &= \text{AB-C-BDAB} \\
y' &= \text{-BDCAB-A-}
\end{align*}
\]

What is the complexity of the naive algorithm? How to make this efficient?
Recursive Approach

\(LCS(i, j)\) means length of LCS considering only up to \(x_i\) and \(y_j\)
**Recursive Approach**

$LCS(i, j)$ means length of LCS considering only up to $x_i$ and $y_j$

\[
LCS(i, j) = \begin{cases} 
0 & \text{if } i \text{ or } j = 0 \\
LCS(i - 1, j - 1) + 1 & \text{if } x_i = y_j \\
\max(LCS(i - 1, j), LCS(i, j - 1)) & \text{otherwise}
\end{cases}
\]
global optimum uses optimal solutions of subproblems

Proof by contradiction: What if subsolution were not optimal?

Let \( z \) be an optimal LCS of \( i, j \) of length \( k \).

1. If \( x_i = y_j \), then \( z_k = x_i = y_j \) and
   \[
   LCS(i - 1, j - 1) = z_0...z_{k-1}.
   \]
   Not including \( z_k \) makes LCS suboptimal: contradiction!
   If \( z_0...z_{k-1} \) were not LCS, \( z \) could be longer, hence not optimal: contradiction!

2. If \( x_i \neq y_j \) and \( z_k \neq x_i \), then \( z \) is LCS of \( i - 1, j \).
   If longer exists, \( z \) would not be an LCS: contradiction!

3. If \( x_i \neq y_j \) and \( z_k \neq y_j \), then \( z \) is LCS of \( i, j - 1 \)
   Similar to 2.
Summary of Dynamic Programming

1. optimal substructure: global optimum uses optimal solutions of subproblems
2. ordering over subproblems: solve ‘smallest’ first, build ‘larger’ from them
3. ‘overlapping’ subproblems: polynomial number of subproblems, each possibly used multiple times
4. independent subproblems: optimal solution of one subproblem doesn’t affect optimality of another

- top-down: memoization
- bottom-up: compute table, then recover solution
For example:

- What’s still confusing?
- What question didn’t you get to ask today?
- What would you like to hear more about?

Please write down your most pressing question about algorithms and put it in the box on your way out.

Thanks!