http://www.cs.unh.edu/~ruml/cs758

2 handouts: slides, asst 4
Please write down precise pseudo-code for inserting a fresh node $z$ into a binary search tree rooted at $t$. You may assume that the regular comparison predicates ($\approx$, $<$, $\ldots$) work on $z$.key and the keys of the nodes in the tree.
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<td>Binary tree (balanced)</td>
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Balanced Trees

1. AVL Trees (1962)
2. 2-3 Trees
3. red-black trees (1972, popularized 1978)
4. AA trees (1992)
5. left-leaning red-black trees (2008)

probabilistically balanced

1. treaps
2. skip lists
node: data, left, right, parent, color

1. every node is either red or black
2. the root is black
3. (consider nil to be black)
4. both children of a red node are black
5. from any node, all paths to leaves have the same ‘black height’

search and traversal are unchanged
useful subroutines:

- rotate-right
- rotate-left
1. \( n \)'s parent \( \leftarrow \) find-parent(\( n \), root, nil)  
2. if parent is nil  
3. root \( \leftarrow n \)  
4. else  
5. if \( n \) should be before parent  
6. parent's left child \( \leftarrow n \)  
7. else  
8. parent's right child \( \leftarrow n \)  
9. \( n \)'s children \( \leftarrow \) nil
1. \( n \)’s parent \( \leftarrow \) find-parent\((n, \text{root}, \text{nil})\)
2. if parent is nil
3. \( \text{root} \leftarrow n \)
4. else
5. if \( n \) should be before parent
6. \( \text{parent’s left child} \leftarrow n \)
7. else
8. \( \text{parent’s right child} \leftarrow n \)
9. \( n \)’s children \( \leftarrow \text{nil} \)
10. color \( n \) red
11. fix-insert\((n)\)
Recall properties:

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5. from any node, all paths to leaves have the same 'black height'
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Fixing Insertion

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Cases:

1. red root (property 2)
2. two red in a row (property 4)
Fixup Invariant

Cases:
1. red root (property 2)
2. two red in a row (property 4)

During fixup:
1. \( z \) is red
2. if \( z \)’s parent is the root, it is black
3. at most, property 2 xor 4 is violated at \( z \)
   (a) if 2: because \( z \) is root and red
   (b) if 4: because \( z \) and parent are red
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Initialization:
1. we colored \( z \) red
2. we didn’t touch \( z \)’s parent, and roots are black
3. just saw this
**Fix-insert(\(z\))**

1. while \(z\)'s parent is red
2. if \(z\)'s parent is a left child
3. \(y \leftarrow z\)'s uncle (a right child)
4. if \(y\) is red
5. color \(z\)'s parent black \(\text{case 1}\)
6. color \(z\)'s uncle \(y\) black
7. color \(z\)'s grandparent red
8. \(z \leftarrow z\)'s grandparent
9. else if \(z\) is a right child
10. \(z \leftarrow z\)'s parent \(\text{case 2}\)
11. rotate-left(\(z\))
12. color \(z\)'s parent black \(\text{case 3}\)
13. color \(z\)'s grandparent red
14. rotate-right(\(z\)'s grandparent)
15. else, 3 symmetric cases (left↔right)
16. color root black
Assuming other properties are maintained, are we red-black now?

Leverage the invariant:

1. irrelevant
2. irrelevant
3. only 2 xor 4 can be violated in loop
   (a) if 2: root colored black at end, so 2 not violated
   (b) if 4: z’s parent now black, so 4 not violated
Assuming other properties are maintained, are we red-black now?

Leverage the invariant:

1. irrelevant
2. irrelevant
3. only 2 xor 4 can be violated in loop
   (a) if 2: root colored black at end, so 2 not violated
   (b) if 4: \( z \)'s parent now black, so 4 not violated

What about maintaining the other red-black properties?
Red-Black Trees

Maintenace
Case 1
Case 2
Case 3
Complexity
EOLQs

Red-Black Trees
central problem: prop 4 violated: $z$ and parent are red

3 cases (+ 3 more by symmetry of $z$’s parent being left/right):

1. $z$’s uncle $y$ is also red (we have a red layer)
2. $z$’s uncle $y$ is black and $z$ is right child
3. $z$’s uncle $y$ is black and $z$ is left child
central problem: prop 4 violated: \( z \) and parent are red

3 cases (+ 3 more by symmetry of \( z \)’s parent being left/right):
1. \( z \)’s uncle \( y \) is also red (we have a red layer)
2. \( z \)’s uncle \( y \) is black and \( z \) is right child
3. \( z \)’s uncle \( y \) is black and \( z \) is left child

Plan:
1. fix case 1, possibly introducing case 2.
2. reduce case 2 to case 3.
3. fix case 3.
Case 1

case 1: \( z \)'s uncle \( y \) is also red

solution: move redness up

1. color \( z \)'s parent and uncle black
2. color grandparent red and recur

fixup loop invariants:

1. \( z \) is red
2. if \( z \)'s parent is the root, it is black (unchanged)
3. at most, property 2 xor 4 is violated at new \( z \). Note previous violations at old \( z \) are fixed.
   
   (a) if 2: because \( z \) is root and red
   (b) if 4: because \( z \) and parent are red

if new \( z \) is root, will be colored black, increasing all heights
case 2: \( z \)'s uncle \( y \) is black and \( z \) is right child

reduce to case 3: \( z \)'s uncle \( y \) is black and \( z \) is left child

rotation doesn't affect any properties
Case 3

case 3: z’s uncle y is black and z is left child

fix prop 4 at z: pull blackness down to z’s parent and rotate grandparent under it.

fixup loop invariants:

1. z is red
2. if z’s parent is the root, it is black
3. at most, property 2 xor 4 is violated at z.
   (a) can’t be prop 2
   (b) if 4: fixed because z’s parent is now black
   (c) note black-height is preserved!

We are done and loop will exit
finding place is
finding place is $O(\lg n)$

one fixup iteration is constant time

fixup loops only when moving up, so is
finding place is $O(\lg n)$

one fixup iteration is constant time

fixup loops only when moving up, so is $O(\lg n)$

how many rotations are performed?
EOLQs

For example:

- What’s still confusing?
- What question didn’t you get to ask today?
- What would you like to hear more about?

Please write down your most pressing question about algorithms and put it in the box on your way out.

*Thanks!*