http://www.cs.unh.edu/~ruml/cs758
NP-Completeness
optimization vs decision: if opt were easy, decision would be too

P: solvable in polynomial time

NP: \( \exists \) certificate verifiable in polynomial time

NP-Hard: as hard as any problem in NP (via polytime reduction)

NP-Complete: NP-Hard and in NP

reduce \( a \) to \( b \): \( a \rightarrow b \) in polytime, decide \( b \)

\( b \) hard by reduction from \( a \): if \( a \rightarrow b \) in polytime and \( b \) polytime, could solve \( a \)
The Power of Reduction

Theorem: If $B \leq_P A$ for some $B \in \text{NPC}$, then $A$ is NP-Hard.

Since $B$ is NPC, we have $\forall C \in \text{NP}, C \leq_P B$. Since $B \leq_P A$, then $C \leq_P A$ which shows $A$ is NP-Hard.

If also $A \in \text{NP}$, then since $A \in \text{NP}$, we have $A \in \text{NPC}$. 
Reductions

NP-Completeness
- Terms
- Interchangability
- Reductions
  - NPC Proofs
  - C-SAT is in NP
  - C-SAT is NP-Hard
  - Break

SAT

CIRCUIT-SAT

SAT

3-CNF SAT

CLIQUE

SUBSET-SUM

VERTEX-COVER

HAM-CYCLE

TSP
To prove some problem $A$ is NP-Complete:

1. Prove $A \in NP$
2. Pick a known NP-Complete problem $B$
3. Find a translation of instances of $B$ into instances of $A$
4. Show that translated $A$ version is accepted if and only if the original $B$ version should be accepted.
5. Prove that the reduction runs in polynomial time.
Circuit-SAT is in NP

Circuit-SAT: is circuit satisfiable? (otherwise, can be removed)

Certificate is value for every wire.
Simply check that each gate is computed correctly and output is true.
Need to construct reduction \( f \) from any \( L \in \text{NP} \). Given input \( x \in L \), resulting circuit \( C \in \text{Circuit-SAT} \) iff \( x \in L \). We’ll make \( C \) so it’s \( \text{SAT} \) iff \( \exists y \) s.t. verification algorithm \( A(x, y) \) for \( L \) gives true. Intuition: for input \( y \), run \( A(x, y) \).

Let \( n = |x| \) and \( T(n) = O(n^k) \) be bound on \( A \)'s running time. Let \( M \) be a circuit for a stored-program computer (including PC and storage). String \( T(n) \) of them together to form \( C'' \). \( C \) is \( C'' \) with input hardwired to program for \( A \) and input \( x \), and output hardwired to result of \( A \). Input to \( C \) is \( y \).

Iff \( y \) exists, \( C \) is satisfiable, so we have a reduction. \( A \) is constant size and uses poly storage. \( M \) is poly size and needs poly steps to run \( A \). \( y \) is poly sized. So \( C'' \) and \( C \) have size polynomial in \( n \) and can be constructed in polynomial time.
### Break

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NP-Completeness

**SAT**
- NPC Proofs
- Reduction
- 3-CNF SAT
- Reductions
- EOLQs

**SAT**
To prove some problem $A$ is NP-Complete:

1. Prove $A \in \text{NP}$
2. Pick a known NP-Complete problem $B$
3. Find a translation of instances of $B$ into instances of $A$
4. Show that translated $A$ version is accepted if and only if the original $B$ version should be accepted.
5. Prove that the reduction runs in polynomial time.
Consider formula with $n$ variables and $m$ connectives.

**SAT ∈ NP:** given variables assignments, evaluate formula.

**SAT is NP-Hard:** Reduction from Circuit-SAT. Basic translation fails on shared subcircuits.
Instead, use one variable for each wire and one clause per gate. Combine clauses with $\land$ and include $\land x_0$ (output).
SAT iff wires in circuit have legal values yielding true.
CNF where each clause has exactly 3 literals. Aka 3-SAT.

3-CNF SAT ∈ NP: given variables assignments, evaluate formula.

3-CNF SAT is NP-Hard: Reduction from SAT. Construct expression tree and convert to binary branching. Assign each node a variable. Form clause for each internal node’s variable, e.g.: \( y_3 \leftrightarrow (y_1 \lor y_2) \)

Clauses will have at most 3 literals. Convert each clause to CNF: form complete truth table, form DNF for false rows, negate and push \( \neg \) inward (using DeMorgan) to get CNF

For each binary clause \((l_1 \lor l_2)\), convert to 
\((l_1 \lor l_2 \lor p) \land (l_1 \lor l_2 \lor \neg p)\).

For each unit clause \((l)\), convert to
\((l \lor p \lor q) \land (l \lor p \lor \neg q) \land (l \lor \neg p \lor q) \land (l \lor \neg p \lor \neg q)\).

Each step preserves satisfiability and is polynomial time.
Reductions

NP-Completeness

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CIRCUIT-SAT
\[\rightarrow\]
 SAT
\[\rightarrow\]
 3-CN SAT
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 VERTEX-COVER
\[\rightarrow\]
 HAM-CYCLE
\[\rightarrow\]
 TSP
For example:

- What’s still confusing?
- What question didn’t you get to ask today?
- What would you like to hear more about?

Please write down your most pressing question about algorithms and put it in the box on your way out.

*Thanks!*